Definitions: von Neumann entropy. *In this series we will derive some useful properties of the von Neumann entropy: the quantum version of Shannon entropy.*

The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathscr{H}_A)$ is defined as

$$H(A)_\rho = -\operatorname{tr}\left(\rho \log \rho\right) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.

Given a composite system $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$ we write $H(AB)_\rho$ to denote the entropy of the reduced state of a subsystem, $\rho_{AB} = \operatorname{tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we drop the indices.

The conditional *von Neumann entropy is defined as*

$$H(A|B)_\rho = H(AB)_\rho - H(B)_\rho. \tag{2}$$

In the Alice-and-Bob picture this quantifies the uncertainty that Bob (who holds the B part of the quantum state ρ_{AB}) still has about Alice's state.

The strong sub-additivity property of the von Neumann entropy is very useful. It applies to a tripartite composite system $\mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$,

$$H(A|BC)_\rho \leq H(A|B)_\rho. \tag{3}$$

Tips: Handy properties of von Neumann entropy

1. Definition: $H(A)_\rho = -\operatorname{tr}\left(\rho \log \rho\right) = -\sum_i \lambda_i \log \lambda_i$, where:
   (a) $\{\lambda_i\}_i$ are the eigenvalues of ρ;
   (b) the logarithm is $\log_2$;
   (c) to evaluate the entropies, $0 \log 0 = 0$;
   (d) nonation: we sometimes see just $H(A)$ or even $H(\rho)$.
2. Positivity: $H(A)_\rho \geq 0$ (because $0 \leq \lambda_i \leq 1$).
3. Entropy of pure states: $H(A)_{|\psi\rangle} = 0$ (because the density matrix has a single eigenvalue 1 for eigenvector $|\psi\rangle$).
4. Basis independence: $H(A)_\rho = H(A)_{U\rho U^\dagger}$ for unitaries U, because the eigenvalues are not affected by a change of basis.
5. Conditional entropy: $H(A|B)_\rho = H(AB)_\rho - H(B)_\rho$.
6. Strong subadditivity: $H(A|BC)_\rho \leq H(A|B)_\rho$. In other words, knowing more cannot hurt.

Exercise 1. *Properties of the von Neumann Entropy.*

(a) Prove the following general properties of the von Neumann entropy:

(i) $H(A)_\rho \geqslant 0$ for any ρ_A.

(ii) *If ρ_{AB} is pure, then $H(A)_\rho = H(B)_\rho$.*
Hint. Use the Schmidt decomposition of bipartite pure states: for any $|\psi\rangle_{AB}$, there exist coefficients p_k, and two orthonormal sets of vectors $\{|\chi_k\rangle_A\}_k$ and $\{|\phi_k\rangle_B\}_k$, such that $|\psi\rangle_{AB} = \sum_k \sqrt{p_k}\, |\chi_k\rangle_A \otimes |\phi_k\rangle_B$.

(iii) *If two systems are independent, $\rho_{AB} = \rho_A \otimes \rho_B$, then $H(AB)_\rho = H(A)_{\rho_A} + H(B)_{\rho_B}$.*

(b) *Consider a bipartite state that is classical on subsystem Z: $\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z$ for some basis $\{|z\rangle_Z\}_z$ of $\mathscr{H}_Z$. Show that:*

(i) *The conditional entropy of the quantum part, A, given the classical information Z is*

$$H(A|Z)_\rho = \sum_z p_z\, H(A|Z=z), \tag{4}$$

where $H(A|Z=z) = H(A)_{\rho_A^z}$.

(ii) *The entropy of A is concave,*

$$H(A)_\rho \geq \sum_z p_z\, H(A|Z=z). \tag{5}$$

(iii) *The entropy of a classical probability distribution $\{p_z\}_z$ cannot be negative, even if one has access to extra quantum information, A,*

$$H(Z|A)_\rho \geq 0. \tag{6}$$

Remark: Eq (6) holds in general only for classical Z. Bell states are immediate counterexamples in the fully quantum case.

Tips— In this exercise you have to prove some more properties of von Neumann entropy. The first one is rather surprising: if two systems share a pure state, then the entropy of each of the systems is the same, independently of their dimensions. In other words, if you have a pure state $|\psi\rangle$ in a system represented by the hilbert space $\mathscr{H}$, then you can decompose the system in two parts, $\mathscr{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, in any way you want and the entropy of A will always be equal to the entropy of B, even if you choose to split $\mathscr{H}$ in a way such that $|\mathscr{H}_A| \ll |\mathscr{H}_B|$.

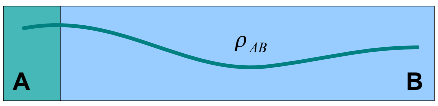

Figure 1: If ρ_{AB} is pure $H(A)_\rho = H(B)_\rho$, independently of dimensions of subsystems A and B.

To prove this, try writting a Schmidt decomposition of $|\psi\rangle$ (page 27 of the script).

The next property studies two systems that are in a product state, $\rho_{AB} = \rho_A \otimes \rho_B$. The systems are independent of each other—whatever operations or measurements you perform on A will not affect ρ_B and vice-versa. In this non-correlated case one would expect that the uncertainty about the global state is just the sum of the uncertainty about the two local subsystems—and, for once, quantum mechanics respects common sense, with $H(AB) = H(A) + H(B)$.

To prove that property, you may start by expanding the reduced states in their eigenbases,

$$\rho_A = \sum_k \gamma_k |k\rangle\langle k|_A, \qquad \rho_B = \sum_\ell \lambda_\ell |\ell\rangle\langle \ell|_B.$$

Now expand the composed state $\rho_{AB} = \rho_A \otimes \rho_B$ in those bases and compute its entropy directly.

In part *b*) we look at a special category of bipartite states, those that are classical on one of the subsystems. These states are introduced on pages 34–35 of the script. They have the form

$$\rho_{ZA} = \sum_z p_z |z\rangle\langle z|_Z \otimes \rho_A^z \tag{T.1}$$

for a fixed basis $\{z_Z\}_z$ of the first subsystem $\mathscr{H}_Z$ and a probability distribution $\{p_z\}_z$.

It help to look at one example of such a state. Consider two qubits, the computational basis and the classicaly correlated state

$$\rho_{ZA} = p\ |0\rangle\langle 0| \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + (1-p)\ |1\rangle\langle 1| \otimes \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

Actually, the first system can be a classical bit, since no cross terms like $|0\rangle\langle 1|$ appear there. The reduced state of system A is just

$$\rho_A = \mathrm{tr}_Z(\rho_{ZA}) = p\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + (1-p)\ \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix},$$

and in general, for a hybrid classical-quantum state of the form of Eq. T.1,

$$\rho_A = \sum_z p_z\ \rho_A^z.$$

The reduced state of the classical system is

$$\rho_Z = \mathrm{tr}_A(\rho_{ZA}) = p(\alpha+\delta)\ |0\rangle\langle 0| + (1-p)(\alpha'+\delta')\ |1\rangle\langle 1|,$$

or, in general,

$$\rho_z = \sum_z p_z\, \mathrm{tr}(\rho_A^z)\ |z\rangle\langle z|.$$

These hybrid states may be interpreted as "state ρ_A^z was prepared on system A with probability p_z, and in that case the classical register Z shows the value z, i.e., it is in the pure state $|z\rangle$." A measurement on system Z performed in basis $\{z\}_z$ would allow us to determine which ρ_A^z had been prepared, because the total state would became $|z\rangle\langle z| \otimes \rho_A^z$. Since in that case the reduced state of A would be ρ_A^z, we call that "the state of system A conditioned on the measurement outcome z of system Z", $\rho_A^z = \rho_{A|Z=z}$.

Let us now go back to the exercise. You are asked to prove that for states like that of Eq. T.1,

$$\begin{aligned} H(AZ) &= H(Z)_\rho + \sum_z p_z\ H(A|Z=z) \\ &= H(Z)_\rho + \sum_z p_z\ H(A)_{\rho_A^z}. \end{aligned}$$

I suggest that you expand the matrices ρ_A^z in their eigenbases, for instance

$$\rho_A^z = \sum_k \lambda_k^z\ |k_z\rangle\langle k_z|.$$

If you now write ρ_{ZA} using those expressions for ρ_A^z and compute its entropy, you should get the desired result.

I won't help you in part (b) (ii). Part (b) (iii) asks you to show that for these states $H(Z|A) \geq 0$. One trick that may help is to imagine a system Y that is just a copy of Z and a state

$$\rho_{ZAY} = \sum_k p_z |z\rangle\langle z| \otimes \rho_A^z \otimes |y\rangle\langle y|.$$

You may check that the entropy of this state is the same than that of ρ_{AB}. In fact, you can show that $H(ZAY) = H(ZA)$ and $H(Z) = H(Y)$. Now use strong subaddivity to show what you want.

Exercise 2. ***Von Neumann Entropy and Entanglement.***

(a) Compute the entropies $H(A)$, $H(AB)$ and the conditional entropy $H(A|B)$ of the Bell state

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|00\rangle_{AB} + |11\rangle_{AB}\right) \; ; \tag{7}$$

(b) Calculate the conditional entropies $H(A|BC)$, $H(AB|C)$ and $H(A|B)$ of the GHZ state

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} \left(|000\rangle_{ABC} + |111\rangle_{ABC}\right) \; . \tag{8}$$

Consider a separable state ρ_{AB}, i.e. a state that can be written as a convex combination of product states:

$$\rho_{AB} = \sum_k p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)} \; , \tag{9}$$

where $\{p_k\}_k$ is a probability distribution.

(c) Prove that the von Neumann entropy is always positive for such a state,

$$H\left(A|B\right)_\rho \geqslant 0 \; . \tag{10}$$

Remark: This means that, whenever the conditional entropy is negative, you are necessarily in possession of an entangled state.

Hint. First use the results of point (b) of the previous exercise to prove that the conditional von Neumann entropy is concave, i.e. if $\rho_{AB} = \sum_k p_k \, \rho_{AB}^{(k)}$, then

$$H\left(A|B\right)_\rho \geqslant \sum_k p_k \, H\left(A|B\right)_{\rho^{(k)}} \; . \tag{11}$$