Exercise 1. ***Measurements and dephasing***

We will see how to write measurements using generalized Pauli operators. This helps us prove the uncertainty relation (1).

We saw that a mesurement described by a POVM $\{M_k\}_k$ corresponds to the following unitary evolution of the quantum system measured and a classical register,

$$|\text{ready to measure}\rangle\langle\text{ready to measure}| \otimes \rho \mapsto \sum_k |k\rangle\langle k| \otimes \sqrt{M_k}\rho\sqrt{M_k}^\dagger.$$

Tracing out the register, we obtain the map

$$\rho \mapsto \sum_k \sqrt{M_k}\rho\sqrt{M_k}^\dagger$$

on the quantum system.

(a) We start with the measurement of a single qubit. A measurement in the computational basis corresponds to the POVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and the evolution

$$\rho \mapsto |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|.$$

Show that the map above can also be written as

$$\rho \mapsto \frac{1}{2}\left(\rho + Z\rho Z^{-1}\right),$$

where Z is the Pauli-Z operator,

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(b) More generally, the measurement of a system of dimension d (like $\log d$ qubits) in an orthonormal basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\}$ can be described as a map

$$\rho \mapsto \sum_{k=0}^{d-1} |k\rangle\langle k|\rho|k\rangle\langle k|.$$

Show that this map can also be expressed as

$$\rho \mapsto \frac{1}{d}\sum_{k=0}^{d-1} Z^k \rho Z^{-k},$$

where Z is the generalized Pauli-Z operator,

$$Z = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}k}|k\rangle\langle k| = \begin{bmatrix} 1 & & & \\ & e^{\frac{2\pi i}{d}} & & \\ & & \ddots & \\ & & & e^{(d-1)\frac{2\pi i}{d}} \end{bmatrix}.$$

Note: use $Z^0 = \mathbb{1}$.

(c) Now we will see how to express a measurement in the complementary basis, $\{|\bar{k}\rangle\}$, with $|\bar{k}\rangle = F|k\rangle$. Here, F is the quantum Fourier transform, whose matrix representation is

$$F = \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} e^{\frac{2\pi i}{d}(jk)}|k\rangle\langle j|.$$

Note that F is unitary ($F^\dagger = F^{-1}$). A measurement in this basis corresponds to the map

$$\rho \mapsto \sum_{k=0}^{d-1} |\bar{k}\rangle\langle\bar{k}|\rho|\bar{k}\rangle\langle\bar{k}| = \sum_{k=0}^{d-1} \left(F|k\rangle\langle k|F^\dagger\right) \ \rho \ \left(F|k\rangle\langle k|F^\dagger\right).$$

Consider the generalized Pauli-X operator,

$$X = \left(\sum_{k=1}^{d} |k-1\rangle\langle k|\right) + |d-1\rangle\langle 0| = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}.$$

Show that we can write a measurement in the complementary basis as

$$\rho \mapsto \frac{1}{d}\sum_{k=0}^{d-1} X^k \rho X^{-k}.$$

Hint: start by showing that $X^k = FZ^kF^\dagger$.

Exercise 2. ***Entropic uncertainty relations***

In the lecture, we proved an entropic uncertainty relation for complementary observables X and Z on a single qubit C, conditioned on side information C.

(a) Generalize the proof of the uncertainty relation for an abritrary quantum system C, and complementary observables X and Z, obtaining

$$H(X|D) + H(Z|D) \geq \log|C| + H(C|D). \tag{1}$$

Hint: follow the proof from the lecture, and use the result from Exercise 1.

(b) Use this relation to prove the Maassen-Uffink relation for complementary observables,

$$H(X) + H(Z) \geq \log|C|. \tag{2}$$

Exercise 3. ***The uncertainty game***

We will see a "practical" application of the entropic uncertainty relation (1). Consider the following setting: two players, Alice and Bob, sit in separate labs. Bob sends one qubit to Alice, she performs one of two measurements (X or Z) at random, and then tells Bob her choice of measurement. Now Bob has to guess the output of Alice's measurement.

(a) Suppose that Bob only has a classical memory (like a notepad) in his lab. What is his best strategy? What is the state of the qubit he should send to Alice, and what is his uncertainty about the measurement outcome?

(b) Now imagine that Bob has a quantum memory: he can prepare two qubits in any state, send one to Alice and keep the other. What should he do now, and how frequently will he guess correctly?