Exercise 1. ***Composing Systems: The Tensor Product.***

You have learned from quantum mechanics that the composition of two systems described by states $|\psi_A\rangle \in \mathscr{H}_A$ and $|\psi_B\rangle \in \mathscr{H}_B$ is described by a state in the tensor product space $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \in \mathscr{H}_A \otimes \mathscr{H}_B$. The tensor product space is defined by its basis elements: if $\{|\phi_A^i\rangle\}$ and $\{|\phi_B^j\rangle\}$ are bases of $\mathscr{H}_A$ and $\mathscr{H}_B$, respectively, then

$$\mathscr{H}_A \otimes \mathscr{H}_B = \mathrm{span}\left\{|\phi_A^i\rangle \otimes |\phi_B^j\rangle\right\} \ . \tag{1}$$

The tensor product satisfies the following basic properties:

$$\big(|\psi_A\rangle + |\psi'_A\rangle\big) \otimes |\psi_B\rangle = |\psi_A\rangle \otimes |\psi_B\rangle + |\psi'_A\rangle \otimes |\psi_B\rangle \ ; \tag{2}$$

$$(\alpha|\psi_A\rangle) \otimes |\psi_B\rangle = \alpha \cdot \ |\psi_A\rangle \otimes |\psi_B\rangle \ , \tag{3}$$

and the same properties hold on the second term.

(a) Consider two qubits, $\mathscr{H}_A = \mathscr{H}_B = \mathbb{C}^2$, with respective bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. The tensor product space admits the basis $\{|0_A\rangle\otimes|0_B\rangle, |0_A\rangle\otimes|1_B\rangle, |1_A\rangle\otimes|0_B\rangle, |1_A\rangle\otimes|1_B\rangle\}$. Write the state of each of the following systems in this basis.

(i) System A in state $|0_A\rangle$ and system B in state $|1_B\rangle$.

(ii) System A in state $\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)$ and system B in state $|1_B\rangle$.

(iii) System A in state $\frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)$ and system B in state $\frac{1}{\sqrt{2}}(|0_B\rangle + |1_B\rangle)$.

An important property of the tensor product space is that there are states in $\mathscr{H}_A \otimes \mathscr{H}_B$ which cannot themselves be written as a tensor product of states from each space, i.e. they cannot be written in the form $|\psi_A\rangle \otimes |\psi_B\rangle$.

(b) Consider two qubits, $\mathscr{H}_A = \mathscr{H}_B = \mathbb{C}^2$, with respective bases $\{|0_A\rangle, |1_A\rangle\}$ and $\{|0_B\rangle, |1_B\rangle\}$. Consider the state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0_A\rangle \otimes |0_B\rangle + |1_A\rangle \otimes |1_B\rangle\right) \ . \tag{4}$$

Show that this state vector cannot be written as a tensor product of two individual state vectors in $\mathscr{H}_A$ and $\mathscr{H}_B$.

(c) Show that the state given in (b) can be written as

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|+_A\rangle \otimes |+_B\rangle + |-_A\rangle \otimes |-_B\rangle\right) \ , \tag{5}$$

with $|\pm_A\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle + |1_A\rangle)$

(d) *(Extra question, with an introduction to tomography.)* We have shown that no state vector can appropriately describe the system A from point (b). However, it can be described by a density operator. Determine the density operator ρ for that system by considering explicitely the probabilities of the outcomes of the measurements in the bases $\{|0_A\rangle, |1_A\rangle\}$, $\{|+_A\rangle, |-_A\rangle\}$, and $\{|+i_A\rangle, |-i_A\rangle\}$ (where $|\pm i_A\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle \pm i|1_A\rangle)$).

Hints. By "measuring in a specific basis", it is meant to measure an observable that is diagonal in that basis. Recall also that the probability for measuring the outcome $|\phi\rangle$ is given by $\langle\phi|\rho|\phi\rangle$.

In the density operator formalism, everything stays the same: the composition of two systems described by density operators $\rho_A \in \mathcal{S}(\mathscr{H}_A)$ and $\rho_B \in \mathcal{S}(\mathscr{H}_B)$ respectively, is described by a density operator $\rho \in \mathcal{S}(\mathscr{H}_A \otimes \mathscr{H}_B) = \mathcal{S}(\mathscr{H}_A) \otimes \mathcal{S}(\mathscr{H}_B)$. The important difference, however, is that ρ is not necessarily $\rho_A \otimes \rho_B$. Moreover, in contrast to state vectors, whatever the state of the joint system is, one can always write down the density operator of one part of the joint system, called *reduced state* or *marginal state*. The reduced state is obtained by *partial trace.*

(e) Write out the density operators for the following systems using basis elements of $\mathscr{H}_A \otimes \mathscr{H}_B$ given in point (a). (Use the matrix notation for convenience.)

(i) Two qubits in the state $|\Phi^+\rangle$ defined in point (b).

(ii) Two qubits that are randomly prepared either jointly in state $|0_A\rangle \otimes |0_B\rangle$ or in the joint state $|1_A\rangle \otimes |1_B\rangle$, with probability $1/2$ each.

(iii) *(Greenberger-Horne-Zeilinger state or cat state.)* Three qubits A, B, and C in the state described by the vector

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}} \left(|0_A\rangle \otimes |0_B\rangle \otimes |0_C\rangle + |1_A\rangle \otimes |1_B\rangle \otimes |1_C\rangle\right) \ . \tag{6}$$

(iv) The N-qubit version of the GHZ state,

$$|\mathrm{GHZ}_N\rangle = \frac{1}{\sqrt{2}} \left(|0_1\rangle \otimes \cdots \otimes |0_N\rangle + |1_1\rangle \otimes \cdots \otimes |1_N\rangle\right) \ . \tag{7}$$

(v) The maximally entangled state between two systems A and B, of N qubits each. Let $\{|i_{A/B}\rangle\}_i$ be a basis for each system. The state is given by

$$|\Psi_N\rangle = \frac{1}{\sqrt{2^N}} \sum_i |i_A\rangle |i_B\rangle \ . \tag{8}$$

(f) Calculate the following reduced states from point (e) and give their density operators.

(1.) The reduced state of system (i) on qubit A (respectively on qubit B).

(2.) The reduced state of system (ii) on qubit A (respectively on qubit B).

(3.) The reduced state of the GHZ state (iii) on the two first qubits, A and B.

(4.) The reduced state of the N-qubit GHZ state (iv) on all but the last qubit, i.e. just tracing out the N-th qubit.

(5.) The reduced state of the maximally entangled state $|\Psi_N\rangle$ of point (v) on party A.

Hint. Factorize the state vector cleverly.

(6.) The reduced state of the maximally entangled state $|\Psi_N\rangle$ on the k first qubits of A and B (i.e. tracing out the $N-k$ last qubits of A and B).