Quantum Information Theory
Tips 12

Exercise 12.1 Entropic Uncertainty Relation

For part a), think about extending the space $\mathcal{H}_\mu$, in which S, T and $\tilde{T}$ act, to the space $\mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{H}_\nu$, where $\mathcal{H}_\nu$ is a space isomorphic to $\mathcal{H}_\mu$. Then, use the invariance of D under addition of nullspaces to add on $W := \tilde{T} - T$ to the right-hand side of $D(S||T)$. Finally, consider a map that adds on everything in the second subspace to the first, and use the DPI to prove the desired inequality.
As for part c), describe the X measurement on A with the isometry $V_X = \sum_j |j\rangle \otimes X_j$ and the associated state $\tilde{\rho}_{XABC} = V_X \rho_{ABC} V_X^\dagger$. Then look at this post-measurement-state (remember it is pure!) to transform $H(X|B)$ into a relative entropy between the state $\tilde{\rho}_{XAC}$ and $\mathbb{1}_X \otimes \tilde{\rho}_{AC}$.
In a next step, rewrite the state on the right-hand to the form $V_X(\cdot)V_X^\dagger$, so that you can apply the invariance of D under unitary transformations to go back to ρ_{AC} instead of $\tilde{\rho}_{XAC}$. Then, introduce the Z-measurement and use the DPI. On the right-hand side, this should give you an overlap between the two bases. Finally, use a few more properties of D before transforming back to an expression involving $H(Z|C)$.

Exercise 12.2 Entropic Uncertainty Relation: Examples

In part (a), you basically show that the X and Z measurements are "conjugate," in the sense that the constant $c(X, Z)$ is the same for each projector, and it is also minimal (and therefore the bound in the uncertainty relation $\log 1/c$ is maximal).
In part (b), the state that Alice, Bob, and Charlie share can be described as $|\psi^+\rangle\langle\psi^+| \otimes \rho_C$. To show that Charlie has maximum uncertainty for both the X and the Z measurements, we need that $H(Z|C) \geq 1$ and $H(X|C) \geq 1$. Because this is a cq-state, as you know from a previous exercise sheet, this is maximal, and therefore $H(Z|C) = 1$ and $H(X|C) = 1$. To show the inequalities, we use part (a) and two different versions of the uncertainty relation.

$$H(X|B) + H(Z|C) \geq 1$$
$$H(Z|B) + H(X|C) \geq 1.$$

Now you need to show that $H(X|B) = 0$ and $H(Z|B) = 0$. Given the state that Alice and Bob share, this should be straightforward. Essentially, this means in the end that Charlie cannot find out any information about Alice's measurement outcomes.
For part (c), the first question is quite straightforward from part (b). However, the second question is more nuanced. Try to think about how Charlie's entropies $H(Z|C)$ and $H(X|C)$ could still be maximal, even if Alice and Bob's initial state is pure, but is not maximally entangled. You should show that this is not possible.

Exercise 12.3 Another uncertainty Relation

Remember that ρ_{ABC} is pure!