ETH
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Quantum Information Theory
Solutions 11

HS 12
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Exercise 11.1 Resource inequalities: teleportation and classical communication

We saw a protocol, teleportation, to transmit one qubit using two bits of classical computation and one ebit, $\underset{\Upsilon}{\overset{2}{\rightarrow}} \geq \overset{1}{\rightsquigarrow}$ *(Section 5.1, page 52 of the script). Now suppose that Alice and Bob share unlimited entanglement: they can use up as many ebits as they want. Can Alice send* n *qubits to Bob using less than* $2n$ *bits of classical communication? In other words, we want to know if the following is possible:*

$$\underset{\infty}{\overset{m}{\rightarrow}} \geq \underset{\infty}{\overset{n}{\rightsquigarrow}}, \qquad m < 2n.$$

Prove that this is not the case.

We concatenate teleportation and superdense coding (with unlimited entanglement),

$$\underset{\infty}{\overset{m}{\rightarrow}} \geq \underset{\infty}{\overset{n}{\rightsquigarrow}} \qquad \underset{\infty}{\overset{n'}{\rightsquigarrow}} \geq \underset{\infty}{\overset{m'}{\rightarrow}}.$$

We can fix $n = n'$. Superdense coding allows us to transmit two bits of classical information using one qubit and one ebit, so we have

$$\underset{\infty}{\overset{m}{\rightarrow}} \geq \underset{\infty}{\overset{n}{\rightsquigarrow}} \geq \underset{\infty}{\overset{2n}{\rightarrow}}.$$

Let us focus on the extremes of this resource inequality,

$$\underset{\infty}{\overset{m}{\rightarrow}} \geq \underset{\infty}{\overset{2n}{\rightarrow}}.$$

Our result follows immediately if we assume that $\underset{\infty}{\overset{x}{\rightarrow}} \geq \underset{\infty}{\overset{y}{\rightarrow}}$ implies $y \leq x$ (i.e., entanglement itself does not help us send more classical bits). The proof of that is similar to the one for quantum channels (p. 75 of the script).

Exercise 11.2 A sufficient entanglement criterion

In general it is very hard to determine if a state is entangled or not. In this exercise we will construct a simple entanglement criterion that correctly identifies all entangled states in low dimensions.
Recall that we say that a bipartite state ρ_{AB} *is separable (not entangled) if*

$$\rho = \sum_k p_k \ \sigma_k \otimes \tau_k, \quad \forall k : p_k \geq 0, \sigma_k \in \mathcal{S}_=(\mathcal{H}_A), \tau_k \in \mathcal{S}_=(\mathcal{H}_B), \quad \sum_k p_k = 1.$$

a) *Let* $\Lambda_A : End(\mathcal{H}_A) \mapsto End(\mathcal{H}_A)$ *be a positive map. Show that* $\Lambda_A \otimes \mathcal{I}_B$ *maps separable states to positive operators.*

This means that if we apply $\Lambda_A \otimes \mathcal{I}_B$ *to a bipartite state* ρ_{AB} *and obtain a non-positive operator, we know that* ρ_{AB} *is entangled. In other words, this is a sufficient criterion for entanglement.*

If $\rho \in End(\mathcal{H}_A \otimes \mathcal{H}_B)$ is a separable state, it can be written as convex combination of product states, $\rho = \sum_i p_A \sigma_A^i \otimes \sigma_B^i$, and

$$\Lambda_A \otimes \mathbb{1}_B \left(\sum_i p_A \sigma_A^i \otimes \sigma_B^i \right) = \sum_i p_A \Lambda_A(\sigma_A^i) \otimes \sigma_B^i.$$

All $\left\{ \Lambda_A \left(\sigma_A^i \right) \right\}_i$ are positive operators. Since the set of positive operators is convex, we know that a convex combination of positive operators is still positive.

b) *Now we have to find a suitable map* Λ_A. *Show that the transpose,*

$$\mathcal{T} \left(\sum_{ij} a_{ij} |i\rangle\langle j| \right) = \sum_{ij} a_{ji} \ |i\rangle\langle j|,$$

is a positive map from End($\mathcal{H}_A$) to End($\mathcal{H}_A$), but is not completely positive.

First we show that $\mathcal{T}(AB) = \mathcal{T}(A)\mathcal{T}(B)$,

$$A = \sum_{i,j} a_{ij}\ |i\rangle\langle j|, \qquad B = \sum_{k,\ell} b_{k\ell}\ |k\rangle\langle \ell|, \qquad AB = \sum_{i,j,\ell} a_{ij}b_{j\ell}\ |i\rangle\langle \ell|, \qquad \mathcal{T}(AB) = \sum_{i,j,\ell} a_{ij}b_{j\ell}\ |\ell\rangle\langle i|$$

$$\mathcal{T}(A) = \sum_{i,j} a_{ij}\ |j\rangle\langle i|, \qquad \mathcal{T}(B) = \sum_{k,\ell} b_{k\ell}\ |\ell\rangle\langle k| \qquad \mathcal{T}(B)\mathcal{T}(A) = \sum_{i,j,\ell} a_{ij}b_{j\ell}|\ell\rangle\langle i|$$

We diagonalize the positive operator A as $A = UDU^\dagger$. The positivity of $\mathcal{T}$ follows from

$$\mathcal{T}(A) = \mathcal{T}\left(UDU^\dagger\right) = \mathcal{T}(U^\dagger)\mathcal{T}(D)\mathcal{T}(U) = \mathcal{T}(U^\dagger)D\mathcal{T}(U)$$

But $\mathcal{T}(U^\dagger) = \mathcal{T}(U)^\dagger$,

$$U = \sum_{i,j} u_{ij}|i\rangle\langle j|, \qquad U^\dagger = \sum_{i,j} u_{ij}^*|j\rangle\langle i|, \qquad \mathcal{T}(U) = \sum_{i,j} u_{ij}|j\rangle\langle i|,$$

$$\mathcal{T}(U^\dagger) = \sum_{i,j} u_{ij}^*|i\rangle\langle j|, \qquad \mathcal{T}(U)^\dagger = \sum_{i,j} u_{ij}^*|i\rangle\langle j|.$$

So we have that $\mathcal{T}(A) = \mathcal{T}(U)^\dagger D\mathcal{T}(U)$. If $\mathcal{T}(U)$ is a unitary, then $\mathcal{T}(A)$ has the same eigenvalues as A, since unitary transformations do not change the eigenvalues. Let's check: if U is unitary, we have

$$\mathbb{1} = UU^\dagger = \left(\sum_{i,j} u_{ij}|i\rangle\langle j|\right)\left(\sum_{k,\ell} u_{k\ell}^*|\ell\rangle\langle k|\right) = \sum_{i,j,k} u_{ij}u_{kj}^*|i\rangle\langle k| \ \Rightarrow \sum_j u_{ij}u_{kj}^* = \delta_{ik},$$

$$\mathbb{1} = U^\dagger U = \left(\sum_{k,\ell} u_{k\ell}^*|\ell\rangle\langle k|\right)\left(\sum_{i,j} u_{ij}|i\rangle\langle j|\right) = \sum_{i,j,\ell} u_{i\ell}^*u_{ij}|\ell\rangle\langle j| \ \Rightarrow \sum_i u_{i\ell}^*u_{ij} = \delta_{j\ell}.$$

As for the transpose, we have

$$\mathcal{T}(U)\mathcal{T}(U)^\dagger = \left(\sum_{i,j} u_{ij}|j\rangle\langle i|\right)\left(\sum_{k,\ell} u_{k\ell}^*|k\rangle\langle \ell|\right) = \sum_{j,\ell}\sum_i u_{ij}u_{i\ell}^*|j\rangle\langle \ell| = \mathbb{1},$$

$$\mathcal{T}(U)^\dagger\mathcal{T}(U) = \left(\sum_{k\ell} u_{k\ell}^*|k\rangle\langle \ell|\right)\left(\sum_{i,j} u_{ij}|j\rangle\langle i|\right) = \sum_{i,k}\sum_j u_{kj}^*u_{ij}|k\rangle\langle i| = \mathbb{1},$$

since multiplication is commutative in $\mathbb{C}$. So the transpose of a unitary is a unitary, so the eigenvalues of A are the same as $\mathcal{T}(A)$, so if A is positive so is $\mathcal{T}(A)$, so $\mathcal{T}$ is positive. So.

To see that the transpose is not completely positive, if suffices to apply it to part of a non-normalized maximally entangled state, $|\phi\rangle = \sum_i |i\rangle|i\rangle$ (due to the CJ isomorphism).

$$[\mathcal{I}\otimes\mathcal{T}]\left(\sum_{i,j}|i\rangle|i\rangle\langle j|\langle j|\right) = \sum_{i,j}|i\rangle\langle j|\otimes|j\rangle\langle i|$$
$$= \sum_{i,j}|i\rangle|j\rangle\langle j|\langle i|,$$

which for two qubits, for instance, has the form

$$|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with eigenvalues $\{-1, 1, 1, 1\}$. So, $[\mathcal{I}\otimes\mathcal{T}](|\phi\rangle\langle\phi|)$ is not positive, so $\mathcal{T}$ is not completely positive. Hooray!

c) *Apply the partial transpose, $\mathcal{T}_A \otimes \mathcal{I}_B$, to the ε-noisy Bell state*

$$\rho_{AB}^{\varepsilon} = (1-\varepsilon)\ |\psi^-\rangle\langle\psi^-| + \varepsilon\ \frac{\mathbb{1}_4}{4}, \quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \quad \varepsilon \in [0,1].$$

For what values of ε can we be sure that ρ^{ε} is entangled?

If $[\mathcal{I}\otimes\mathcal{T}](\rho^{\varepsilon})$ is not positive, ρ^{ε} is entangled (note: we have not proved the converse, though it is true too). We have

$$\begin{aligned}
\rho^{\varepsilon} &= \frac{1-\varepsilon}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| - |00\rangle\langle 11| - |11\rangle\langle 00|) + \frac{\varepsilon}{4}(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|) \\
&= \frac{2-\varepsilon}{4}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2-\varepsilon}{4}|1\rangle\langle 1| \otimes |1\rangle\langle 1| - \frac{2-2\varepsilon}{4}|0\rangle\langle 1| \otimes |0\rangle\langle 1| - \frac{2-2\varepsilon}{4}|1\rangle\langle 0| \otimes |1\rangle\langle 0| \\
&\quad + \frac{\varepsilon}{4}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{\varepsilon}{4}|1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
&= \frac{1}{4}\begin{pmatrix} 2-\varepsilon & 0 & 0 & 2-2\varepsilon \\ 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 2-2\varepsilon & 0 & 0 & 2-\varepsilon \end{pmatrix} \\
[\mathcal{I}\otimes\mathcal{T}](\rho^{\varepsilon}) &= \frac{2-\varepsilon}{4}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2-\varepsilon}{4}|1\rangle\langle 1| \otimes |1\rangle\langle 1| - \frac{2-2\varepsilon}{4}|0\rangle\langle 1| \otimes |1\rangle\langle 0| - \frac{2-2\varepsilon}{4}|1\rangle\langle 0| \otimes |0\rangle\langle 1| \\
&\quad + \frac{\varepsilon}{4}|0\rangle\langle 0| \otimes |1\rangle\langle 1| + \frac{\varepsilon}{4}|1\rangle\langle 1| \otimes |0\rangle\langle 0| \\
&= \frac{1}{4}\begin{pmatrix} 2-\varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon & 2-2\varepsilon & 0 \\ 0 & 2-2\varepsilon & \varepsilon & 0 \\ 0 & 0 & 0 & 2-\varepsilon \end{pmatrix}.
\end{aligned}$$

This has eigenvalues $\frac{1}{4}\{2-\varepsilon, 2-\varepsilon, 2-\varepsilon, 3\varepsilon-2\}$. The last eigenvalue is negative for $\varepsilon \le \frac{2}{3}$, so if the state is less than $\frac{2}{3}$-noisy, it is entangled.

Exercise 11.3 Relative Entropy

The quantum relative entropy is defined as $D(\rho||\sigma) = Tr(\rho\log\rho - \rho\log\sigma)$. For two classical probability distributions p and q, this definition simplifies to the expression for the Kullback-Leibler divergence $\sum_j p_j \log\frac{p_j}{q_j}$. Similar to the classical case, the relative entropy serves as a kind of "distance" between quantum states (although it is not technically a metric). Show that

a) $H(A|B) = -D(\rho||\mathbb{1}_A \otimes \rho_B)$.
We can very easily calculate

$$\begin{aligned}
-D(\rho_{AB}||\mathbb{I}_A\otimes\rho_B) &= -\mathrm{Tr}\rho_{AB}(\log\rho_{AB} - \log\mathbb{I}_A\otimes\rho_B) && (1)\\
&= H(AB) - \mathrm{Tr}(\rho_B\log\rho_B) && (2)\\
&= H(AB) - H(B) && (3)\\
&= H(A|B) && (4)
\end{aligned}$$

b) *$D(\rho||\sigma) \ge 0$, with equality if and only if $\rho = \sigma$.*
Write $\rho = \sum_i p_i|i\rangle\langle i|$ and $\sigma = \sum_j q_j|j\rangle\langle j|$. Then $\langle i|\rho = p_i\langle i|$ and

$$\langle i|\log\sigma|i\rangle = \sum_j \log(q_j)P_{ij}, \tag{5}$$

where $P_{ij} := \langle i||j\rangle\langle j||i\rangle \ge 0$, with $\sum_i P_{ij} = 1$ and $\sum_j P_{ij} = 1$. We thus obtain

$$D(\rho||\sigma) = \sum_i p_i\left(\log p_i - \sum_j P_{ij}\log(q_j)\right). \tag{6}$$

Now, since $\log(\cdot)$ is a strictly concave function, we know that $\sum_{ij} P_{ij} \log q_j \leq \log r_i$, where $r_i := \sum_j P_{ij} q_j$, with equality if and only if there is a value for j such that $P_{ij} = 1$. We can now bound the expression for the relative entropy by the classical counterpart

$$D(\rho||\sigma) \geq \sum_i p_i \log \frac{p_i}{r_i}, \tag{7}$$

which can be seen to be non-negative as follows: Using the inequality $\ln x \leq x - 1$, which holds for all positive x with equality if and only if $x = 1$, we get

$$\begin{aligned}
\sum_x p(x) \log \frac{p(x)}{q(x)} &= -\sum_x p(x) \log \frac{q(x)}{p(x)} && (8)\\
&\geq \frac{1}{\ln 2} \sum_x p(x) \left(1 - \frac{q(x)}{p(x)}\right) && (9)\\
&= \frac{1}{\ln 2} \sum_x (p(x) - q(x)) && (10)\\
&= \frac{1}{\ln 2} \sum_x (1 - 1) && (11)\\
&= 0 && (12)
\end{aligned}$$

Looking at the conditions for equality, we find that $D(\rho||\sigma) = 0$ if and only if there is a j for which $P_{ij} = 1$, so that P is a permutation matrix, and $p(x) = q(x)$. As we can relabel the basis vectors for σ if necessary, the conditions obtain precisely if and only if $\rho = \sigma$.

c) *$D(\rho||\sigma) \leq \sum_k p_k D(\rho_k||\sigma)$, where $\rho = p_1\rho_1 + p_2\rho_2$*

$$\begin{aligned}
\sum_k p_k D(\rho_k||\sigma) &= \sum_k p_k (\mathrm{Tr}\rho_k \log \rho_k - \mathrm{Tr}\rho_k \log \sigma) && (13)\\
&= \sum_k p_k (\mathrm{Tr}\rho_k \log \rho_k - \mathrm{Tr}\rho_k \log \rho + \mathrm{Tr}\rho_k \log \rho - \mathrm{Tr}\rho_k \log \sigma) && (14)\\
&= \sum_k p_k (\mathrm{Tr}\rho_k \log \rho_k - \mathrm{Tr}\rho_k \log \rho) + \mathrm{Tr}\rho \log \rho - \mathrm{Tr}\rho \log \sigma) && (15)\\
&= \sum_k p_k D(\rho_k||\rho) + D(\rho||\sigma) && (16)\\
&\geq D(\rho||\sigma) && (17)
\end{aligned}$$

d) *for any CPM $\mathcal{E}$, $D(\rho||\sigma) \geq D(\mathcal{E}(\rho)||\mathcal{E}(\sigma))$*
We split the proof of this data processing inequality into two parts: First, we prove that $D(\rho||\sigma)$ is invariant under isometries, and then that it can only decrease under partial trace operations. This way, because of the Stinespring dilation, we have shown the data processing inequality under general CPMs.

So, first we will calculate:

$$
\begin{aligned}
D(U\rho U^\dagger||U\sigma U^\dagger) &= \mathrm{Tr}(U\rho U^\dagger \log(U\rho U^\dagger) - U\rho U^\dagger \log(U\sigma U^\dagger)) && (18)\\
&= H(\rho) - \mathrm{Tr}(U\rho U^\dagger \log(U\sigma U^\dagger)) && (19)\\
&= H(\rho) - \mathrm{Tr}(\sum_i \lambda_i U|\phi_i\rangle\langle\phi_i|U^\dagger \log(\sum_j s_j U|\psi_j\rangle\langle\psi_j|U^\dagger)) && (20)\\
&= H(\rho) - \mathrm{Tr}(\sum_i \lambda_i |\bar{\phi}_i\rangle\langle\bar{\phi}_i| \log(\sum_j s_j |\bar{\psi}_j\rangle\langle\bar{\psi}_j|)) && (21)\\
&= H(\rho) - \mathrm{Tr}(\sum_{ij} \lambda_i \log(s_j)|\bar{\phi}_i\rangle\langle\bar{\phi}_i|\bar{\psi}_j\rangle\langle\bar{\psi}_j|)) && (22)\\
&= H(\rho) - \mathrm{Tr}(\sum_{ij} \lambda_i \log(s_j)|\langle\bar{\psi}_j|\bar{\phi}_i\rangle|^2) && (23)\\
&= H(\rho) - \mathrm{Tr}(\sum_{ij} \lambda_i \log(s_j)|\langle\psi_j|U^\dagger U|\phi_i\rangle|^2) && (24)\\
&= H(\rho) - \mathrm{Tr}(\sum_{ij} \lambda_i \log(s_j)|\langle\psi_j|\phi_i\rangle|^2) && (25)\\
&= D(\rho||\sigma). && (26)
\end{aligned}
$$

Next, we will show that the relative entropy decreases under partial trace, that is,

$$D(\rho_A||\sigma_A) \leq D(\rho_{AB}||\sigma_{AB}). \tag{27}$$

We find that

$$D(\rho_A||\sigma_A) = D(\rho_A \otimes \frac{\mathbb{1}_B}{d_B}||\sigma_A \otimes \frac{\mathbb{1}_B}{d_B}). \tag{28}$$

Using convexity of the relative entropy together with that there exist unitary transformations U_j on the space B and probabilities p_j such that

$$\rho_A \otimes \frac{\mathbb{1}_B}{d_B} = \sum_j p_j U_j \rho_{AB} U_j^\dagger, \tag{29}$$

we obtain

$$
\begin{aligned}
D(\rho_A||\sigma_A) &= D(\rho_A \otimes \frac{\mathbb{1}_B}{d_B}||\sigma_A \otimes \frac{\mathbb{1}_B}{d_B}) && (30)\\
&\leq \sum_j p_j D(U_j\rho_{AB}U_j^\dagger||U_j\sigma_{AB}U_j^\dagger) && (31)\\
&= \sum_j p_j D(\rho_{AB}||\sigma_{AB}) && (32)\\
&= D(\rho_{AB}||\sigma_{AB}). && (33)
\end{aligned}
$$

e) *$D(\rho||\sigma)$ is not a metric. Do this by showing that it is not symmetric.*
If $D(\rho||\sigma)$ is not symmetric, it is not a metric. And it is easy to see that in general

$$D(\rho||\sigma) = \mathrm{Tr}(\rho\log\rho - \rho\log\sigma) \neq \mathrm{Tr}(\sigma\log\sigma - \sigma\log\rho) = D(\sigma||\rho) \tag{34}$$

is not symmetric (you may construct a specific counterexample if you like).