Quantum Information Theory
Solutions 10

Exercise 10.1 Upper bound on von Neumann entropy

Given a state $\rho \in \mathcal{S}(\mathcal{H}_A)$, show that

$$H(A)_\rho \leq \log |\mathcal{H}_A|. \tag{1}$$

There are several ways to do this. In this exercise, you should do it as follows.
Consider the state $\bar{\rho} = \int U\rho U^\dagger dU$, where the integral is over all unitaries $U \in \mathcal{U}(\mathcal{H})$ and dU is the Haar measure. Find $\bar{\rho}$ and show (1) using concavity:

$$H(A)_\rho \geq \sum_z p_z \, H(A|Z=z). \tag{2}$$

The Haar measure satisfies $d(UV) = d(VU) = dU$, where $V \in \mathcal{U}(\mathcal{H})$ is a unitary.
We use the properties of the Haar measure to verify that $\bar{\rho}$ commutes with all unitaries V on $\mathcal{H}$:

$$V\bar{\rho}V^\dagger = \int (VU)\rho(VU)^\dagger \; dU = \int \tilde{U}\rho\,\tilde{U}^\dagger \; d(V^\dagger \tilde{U}) = \int \tilde{U}\rho\,\tilde{U}^\dagger \; d\tilde{U} = \bar{\rho}.$$

The only density operator on $\mathcal{H}$ that has this property is the completely mixed state: suppose that $\bar{\rho}$ had distinct eigenvalues $\{\lambda_i\}$, and corresponding eigenvectors $\{|i\rangle\}$. Take V to be a unitary transformation that permutes the eigenvectors, for instance $V = |1\rangle\langle 2| + |2\rangle\langle 1|$. Then we would have that $V\bar{\rho}V^\dagger|1\rangle = \lambda_2|1\rangle$, while $\bar{\rho}|1\rangle = \lambda_1|1\rangle$, so $\rho \neq V\rho V^\dagger$. Since all the eigenvalues of $\bar{\rho}$ must be the same, and must be positive and sum up to one, we have that $\bar{\rho} = \mathbb{1}/|\mathcal{H}|$
The concavity property of the von Neumann entropy (Eq. 2) naturally extends to integrals and we get

$$\log|\mathcal{H}| = H\left(\frac{\mathbb{1}}{|\mathcal{H}|}\right) = H(\bar{\rho}) \geq \int H(U\rho\, U^\dagger) \; dU = \int H(\rho) \; dU^{(*)} = H(\rho) \int dU = H(\rho),$$

where $^{(*)}$ stands because the entropy is independent of the basis.

Exercise 10.2 Quantum mutual information

One way of quantifying correlations between two systems A and B is through their mutual information*, defined as*

$$I(A:B)_\rho := H(A)_\rho + H(B)_\rho - H(AB)_\rho \tag{3}$$
$$= H(A)_\rho - H(A|B)_\rho. \tag{4}$$

We can also define a conditional version of the mutual information between A and B as

$$I(A:B|C)_\rho := H(A|C)_\rho + H(B|C)_\rho - H(AB|C)_\rho \tag{5}$$
$$= H(A|C)_\rho - H(A|BC)_\rho. \tag{6}$$

a) *Consider two qubits A and B in a joint state ρ_{AB}.*

1. *Prove that the mutual information of the Bell state state , $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, is maximal. This is why we say Bell states are* maximally entangled
   The global state is pure and the reduced states on A and B are both fully mixed, $\rho_A = \rho_B = \mathbb{1}/2$, so we have
   $$H(AB) = 0, \quad H(A) = H(B) = 1 \quad \Rightarrow \quad I(A:B) = 2,$$
   which is maximal, because the entropy of a single qubit is at most $\log|\mathcal{H}_A| = 1$, as we saw in exercise 9.1 above, and the entropy of the joint state is always non negative.
2. *Show that $I(A:B) \leq 1$ for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|_A \otimes \sigma_B^1$ (where $0 \leq p \leq 1$).*
   We can rewrite the mutual information as
   $$I(A:B) = \underbrace{H(A)}_{\leq 1} - \underbrace{H(A|B)}_{\geq 0^{(*)}} \leq 1$$
   where $^{(*)}$ comes from Lemma 7.5.3 in the script.

b) *Consider the so-called* cat state *shared by four qubits, $A \otimes B \otimes C \otimes D$, that is defined as*

$$\left|\text{🐱}\right\rangle = \frac{1}{\sqrt{2}} \left(|0000\rangle + |1111\rangle \right). \tag{7}$$

Check how the mutual information between qubits A and B changes with the knowledge of the remaining qubits, namely:

1. $I(A:B) = 1$.
2. $I(A:B|C) = 0$.
3. $I(A:B|CD) = 1$.

   How do you interpret these results for the mutual information of the cat state?

The reduced states of the system for k qubits (which are independent of the qubits traced out) have entropies denoted by h_k, given as follows:

$$\begin{aligned}
\rho_4 &= \left|\text{🐱}\right\rangle\left\langle\text{🐱}\right| && \Rightarrow h_4 = 0,\\
\rho_3 &= \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|) && \Rightarrow h_3 = 1,\\
\rho_2 &= \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) && \Rightarrow h_2 = 1,\\
\rho_1 &= \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) && \Rightarrow h_1 = 1.
\end{aligned}$$

The mutual information between A and B given the knowledge of other qubits comes

$$\begin{aligned}
I(A:B) &= H(A) + H(B) - H(AB)\\
&= h_1 + h_1 - h_2 = 1,\\
I(A:B|C) &= H(A|C) + H(B|C) - H(AB|C)\\
&= H(AC) - H(C) + H(BC) - H(C) - H(ABC) + H(C)\\
&= h_2 - h_1 + h_2 - h_1 - h_3 + h_1 = 0,\\
I(A:B|CD) &= H(A|CD) + H(B|CD) - H(AB|CD)\\
&= H(ACD) - H(CD) + H(BCD) - H(CD) - H(ABCD) + H(CD)\\
&= h_3 - h_2 + h_3 - h_2 - h_4 + h_2 = 1.
\end{aligned}$$

These mutual information results can be interpreted as follows. $I(A:B) = 1$ means that upon getting system B, the entropy of A decreases by 1. This makes sense, since A and B are classically correlated, i.e. ρ_{AB} is the quantum representation of a classical probability distribution between A and B, where they share the same bit (either both A and B are 0, or they are both 1).

$I(A:B|C) = 0$ means that if you have system C, the entropy of A does not decrease when you learn B. Again, we have a classically correlated state, and so if you know C, you already know A, and therefore learning B doesn't decrease the entropy of A.

$I(A:B|CD) = 1$ means that if you have system CD, the entropy of A decreases by 1 when you learn B. Now we do not have a classically correlated state, we have a maximally entangled state. If we did have a classically correlated state (such as $1/2(|0000\rangle\langle 0000| + |1111\rangle\langle 1111|)$) then $I(A:B|CD) = 1$, as it was for $I(A:B|C)$. Instead, we have that ρ_{ACD} is classically correlated, but ρ_{ABCD} is entangled. This means that by getting access to B, while already having CD, we now have an entangled state with A, whereas before, we did not. This is why the entropy decreases upon learning B.

Exercise 10.3 Information measures bonanza

Take a system A in state ρ. Non-conditional quantum min- and max-entropies are given by

$$H_{\min}(A)_\rho = -\log \max_{\lambda \in EV(\rho)} \lambda, \qquad H_{\max}(A)_\rho = \log rank(\rho).$$

For instance, if ρ_A has eigenvalues $EV(\rho_A) = \{0.6, 0.2, 0.2, 0\}$, we have $H_{\min}(A)_\rho = -\log 0.6$ and $H_{\max}(A)_\rho = \log 3$.

a) *Show that if $EV(\rho) \prec EV(\tau)$, then the entropy of ρ is larger than or equal to the entropy of τ, for the von Neumann, min- and max-entropies.*

For simplicity, we define again $\vec{r} = \mathrm{EV}(\rho)$ and $\vec{t} = \mathrm{EV}(\tau)$, with the eivenvalues in decreasing order, and the sum-vectors from the previous exercise, $\vec{R} : R_k = \sum_{i=1}^{k} r_i$. We have

$$\vec{r} \prec \vec{t} \Rightarrow r_1 \leq t_1 \Leftrightarrow H_{\min}(A)_\rho \geq H_{\min}(A)_\tau \quad \checkmark$$

$$\vec{R}. \leq \vec{T} \wedge R_n = T_n = 1 \Rightarrow \left|\left\{1\text{'s in } \vec{R}\right\}\right| \leq \left|\left\{1\text{'s in } \vec{T}\right\}\right| \Leftrightarrow |\{0\text{'s in } \vec{r}\}| - 1 \leq \left|\left\{0\text{'s in } \vec{t}\right\}\right| - 1$$
$$\Leftrightarrow \mathrm{rank}(\rho) \geq \mathrm{rank}(\tau) \Leftrightarrow H_{\max}(A)_\rho \geq H_{\max}(A)_\tau \quad \checkmark$$

To prove that the same holds for the von Neumann entropy, we make use of its concavity, $H(A)_{\sum_k p_k \rho_k} \geq \sum_k p_k H(A)_{\rho_k}$. If $\mathrm{EV}(\rho) \prec \mathrm{EV}(\tau)$, we know that there exist $\{U_k, p_k\}_k$ such that $\rho = \sum_k p_k U_k \tau U_k^\dagger$ (Corollary 5.3.3). The von Neumann entropy for state ρ is

$$\begin{aligned} H(A)_\rho &= H(A)_{\sum_k p_k U_k \tau U_k^\dagger} \\ &\geq \sum_k p_k H(A)_{U_k \tau U_k^\dagger} \\ &= \sum_k p_k H(A)_\tau \quad \text{(entropy is invariant under unitaries)} \\ &= H(A)_\tau. \end{aligned}$$

b) *Show that if the bipartite state $|\psi\rangle_{AB}$ can be transformed into $|\phi\rangle$ via LOCC (without catalysts), then $I(A:B)_\psi \geq I(A:B)_\phi$.*

Say $\tau = |\psi\rangle\langle\psi|, \rho = |\phi\rangle\langle\phi|$. Because the global states are pure, $H(AB)_\tau = H(AB)_\rho = 0$, $H(A)_\tau = H(B)_\tau$, and $H(A)_\rho = H(B)_\rho$. If we can go from τ to ρ via LOCC (without catalysts), then $\mathrm{EV}(\tau_A) \prec \mathrm{EV}(\rho_A) \Rightarrow H(A)_\tau \geq H(A)_\rho$. We have

$$I(A:B)_\psi = H(A)_\tau + H(B)_\tau - H(AB)_\tau = 2H(A)_\tau \geq 2H(A)_\rho = I(A:B)_\phi.$$