Quantum Information Theory
Solutions 8

Exercise 8.1 Representations of CPTP maps

Normally, operators that map from the Hilbert space $\mathcal{H}_A$ to $\mathcal{H}_B$ are represented as matrices, such as $C = \sum_{ij} c_{ij} |i\rangle_B \langle j|_A$. We can instead represent them as vectors:

$$|C\rangle\rangle = \sum_{ij} c_{ij} |i\rangle_A |j\rangle_B. \tag{1}$$

We use this notation to denote that the operator C is represented as a vector (and therefore we use a ket), but we use the double right angle bracket to remember that this is an operator and not a state in a Hilbert space. Formally, if $C \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B)$ then $|C\rangle\rangle \in \mathrm{Hom}(\mathbb{C}, \mathcal{H}_A \otimes \mathcal{H}_B)$.

a) *Show that $Y \otimes X\, |Z\rangle\rangle = \left|XZY^T\right\rangle\rangle$, where $X \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B), Y \in \mathrm{Hom}(\mathcal{H}_C, \mathcal{H}_D)$ and $Z \in \mathrm{Hom}(\mathcal{H}_C, \mathcal{H}_A)$. Note that the transpose on Y is defined in the basis chosen to represent the operators in Eq. 1.*

First we write X and Y as

$$X = \sum_{kl} x_{kl} |k\rangle_B \langle l|_A, \qquad Y = \sum_{mn} y_{mn} |m\rangle_D \langle n|_C.$$

Then the LHS is:

$$\begin{aligned} Y \otimes X\, |Z\rangle\rangle &= \sum_{ijklmn} x_{kl} y_{mn} z_{ij} |m\rangle_D \langle n|_C \otimes |k\rangle_B \langle l|_A \left(|j\rangle_C |i\rangle_A\right) \\ &= \sum_{ijkm} x_{ki} y_{mj} z_{ij} |m\rangle_D |k\rangle_B. \end{aligned}$$

The RHS is:

$$\begin{aligned} \left|XZY^T\right\rangle\rangle &= \left| \sum_{ijklmn} x_{kl} y_{mn} z_{ij} |k\rangle_B \langle l|_A |i\rangle_A \langle j|_C |n\rangle_C \langle m|_D \right\rangle\rangle \\ &= \left| \sum_{ijkm} x_{ki} y_{mj} z_{ij} |k\rangle_B \langle m|_D \right\rangle\rangle. \end{aligned}$$

Clearly the LHS is the same as the RHS.

b) *Show that $Tr_A(|X\rangle\rangle \langle\langle Y|) = XY^*$, where $X, Y \in \mathrm{Hom}(\mathcal{H}_A, \mathcal{H}_B)$.*

We can write

$$X = \sum_{kl} x_{kl} |k\rangle_B \langle l|_A, \qquad Y = \sum_{mn} y_{mn} |m\rangle_B \langle n|_A.$$

And so:

$$\begin{aligned} \mathrm{Tr}_A(|X\rangle\rangle \langle\langle Y|) &= \sum_i \langle i|_A \left(\sum_{klmn} x_{kl} y^*_{mn} |l\rangle_A |k\rangle_B \langle n|_A \langle m|_B \right) |i\rangle_A \\ &= \sum_{klm} x_{kl} y^*_{mn} |k\rangle_B \langle m|_B \langle n|_A |l\rangle_A \\ &= AB^*. \end{aligned}$$

c) *We can use the properties (a) and (b) to now derive the Choi-Jamiołkowski representation for CPTP maps. Remember that the operator-sum representation of a map $\mathcal{E} \in \mathrm{Hom}(\mathrm{End}(\mathcal{H}_A), \mathrm{End}(\mathcal{H}_B))$ can be written as:*

$$\mathcal{E}(\rho_A) = \sum_k E_k \rho_A E_k^*,$$

where $\sum_k E_k^* E_k = \mathbb{1}$*. Use (a) and (b) to show that there exists a Choi-Jamiołkowski (CJ) matrix* $T_{A'B} \in \text{End}(\mathcal{H}_{A'B})$ *such that*

$$\mathcal{E}(\rho_A) = Tr_A(T_{AB}(\rho_A^T \otimes \mathbb{1}_B)),$$

where A' *is a copy of* A*, so* $T_{AB} := \sum_{i,j} |i\rangle_A \langle i|_{A'} T_{A'B} |j\rangle_{A'} \langle j|_A$*, and* $\{i\}_i, \{j\}_j$ *are orthonormal bases for both* A *and* A'*.*

Starting with the operator-sum representation, we can write:

$$\begin{aligned}\mathcal{E}(\rho_A) &= \sum_k E_k \rho_A E_k^* \\ &= \sum_k \text{Tr}_A(|\, E_k \rho_A\rangle\rangle \langle\langle E_k|\,) \qquad \text{from (b)} \\ &= \sum_k \text{Tr}_A(\rho_A^T \otimes \mathbb{1}_B \,|\, E_k\rangle\rangle \langle\langle E_k|\,) \qquad \text{from (a)} \\ &= \text{Tr}_A\left((\rho_A^T \otimes \mathbb{1}_B) T_{AB}\right),\end{aligned}$$

where $T_{AB} = \sum_k |\, E_k\rangle\rangle \langle\langle E_k|$. Therefore $T_{A'B} = \sum_{ij} |i\rangle_{A'} \langle i|_A T_{AB} |j\rangle_A \langle j|_{A'}$.

d) *Show that* $T_{A'B}$ *from (c) can also be written as*

$$T_{A'B} = (\mathcal{I}_{A'} \otimes \mathcal{E})(d|\psi^+\rangle_{A'A}\langle\psi^+|),$$

where $|\psi^+\rangle_{A'A} = 1/\sqrt{d} \sum_{i=1}^d |i\rangle_{A'} |i\rangle_A = 1/\sqrt{d}\, |\, \mathbb{1}\rangle\rangle_{A'A}$*.*

We can rewrite this expression as:

$$\begin{aligned}(\mathcal{I}_{A'} \otimes \mathcal{E}_A)(|\, \mathbb{1}\rangle\rangle_{A'A} \langle\langle \mathbb{1}|\,) &= \sum_k \mathbb{1} \otimes E_k \,|\, \mathbb{1}\rangle\rangle_{A'A} \langle\langle \mathbb{1}|\; \mathbb{1} \otimes E_k^* \\ &= \sum_k |\, E_k\rangle\rangle_{A'B} \langle\langle E_k| = T_{A'B}.\end{aligned}$$

e) *What are the CP and TP conditions on* $T_{A'B}$ *in the CJ picture?*

Trace preserving:

$$\text{Tr}(\mathcal{E}(\rho_A)) = \text{Tr}_A(\rho_A^T \text{Tr}_B(T_{AB})) = 1.$$

This has to be true for all ρ_A, and so $\text{Tr}_B(T_{AB}) = \mathbb{1}_A$. This translates directly to $\text{Tr}_B(T_{A'B}) = \mathbb{1}_{A'}$.

Completely Positive:

If $\mathcal{E}$ is CP then

$$\mathcal{I}_C \otimes \mathcal{E}_A(\rho_{CA}) \geq 0,$$

for all systems C. Since from part (d) we know that $T_{AB} = \mathcal{I}_{A'} \otimes \mathcal{E}_A(d|\psi^+\rangle\langle\psi^+|)$ then $T \geq 0$ is a necessary condition for CP. To show that it is sufficient, clearly we have

$$\mathcal{I}_C \otimes \mathcal{E}_A(\rho_{CA}) = \text{Tr}_{CA}(\rho_{CA}^T \otimes \mathbb{1}_{C'B} T_{CAC'B}) \geq 0,$$

where C' is the output system of $\mathcal{I}_C$.

f) *There is another representation called the Normal representation which is defined via the following isomorphism:*

$$\mathcal{N} : \text{Hom}(\text{End}(\mathcal{H}_A), \text{End}(\mathcal{H}_B)) \mapsto \text{Hom}(\text{Hom}(\mathbb{C}, \mathcal{H}_{AA'}), \text{Hom}(\mathbb{C}, \mathcal{H}_{BB'})),$$

where A' *is a copy of* A *and* B' *is a copy of* B*. Show that for any CPTPM* $\mathcal{E}$ *there exists an operator* $N_{AA' \to BB'}$ *such that*

$$N|\rho_A\rangle\rangle = |\mathcal{E}(\rho_A)\rangle\rangle \quad \forall \rho_A \in S_=(\mathcal{H}_A),$$

where A' *is a copy of* A *and* B' *is a copy of* B*.*

This follows from part (a) and the Kraus operator representation of the map $\mathcal{E}$:

$$\begin{aligned} |\,\mathcal{E}(\rho)\rangle\rangle &= \left|\sum_k E_k \rho E_k^*\right\rangle\Bigg\rangle \\ &= \sum_k |\,E_k\rho E_k^*\rangle\rangle \\ &= \sum_k \bar{E}_k \otimes E_k\,|\,\rho\rangle\rangle\,, \end{aligned}$$

where $\bar{E}_k$ indicates the complex conjugate without a transpose. Therefore $N = \sum_k \bar{E}_k \otimes E_k$.

g) *What is the TP condition on N?*

Trace-preserving means that $\mathrm{Tr}(\mathcal{E}(\rho)) = \mathrm{Tr}(\rho)$. First, notice that $\mathrm{Tr}(\rho) = \langle\langle \mathbb{1}|\,|\rho\rangle\rangle$. So we need

$$_{BB'}\langle\langle \mathbb{1}|\,N\,|\rho\rangle\rangle_{AA'} = _{AA'}\langle\langle \mathbb{1}|\,|\rho\rangle\rangle_{AA'}\,.$$

This has to be true for all $\rho_{AA'}$ and therefore

$$_{BB'}\langle\langle \mathbb{1}|\,N\,|\mathbb{1}\rangle\rangle_{AA'} = 1$$

Exercise 8.2 Measurements as unitary evolutions

Consider a measurement on a system $\mathcal{H}_A$, whose output is in $\mathcal{H}_B$ that is described by the observable $O = \sum_{x\in\mathcal{X}} xP_x$, where $\{P_x\}_x$ are projectors. Suppose we want to apply the measurement to the state ρ_A. We can represent the measurement as a unitary evolution on a larger system, followed by a partial trace.

a) *Show that $\mathcal{E}(\rho_A)$ can be written as unitary acting on a larger space $\mathcal{H}\otimes\mathcal{H}$ followed by a partial trace over B. This task can be broken down into the following steps:*

i) *What is the operator-sum representation of the measurement of the operator O?*

We know that the state of the system after measurement is given by $\rho_x = P_x\rho P_x^*/\mathrm{Tr}(\rho P_x)$. Now if we forget the measurement outcome x that we obtained from the measurement, the post-measurement state is described by: $\mathcal{E}(\rho) = \sum_x p(x)\rho_x$. Note that this is an average over the output states weighted by the probability of getting outcome x: $p(x)$. But $p(x)$ is given by $\mathrm{Tr}(\rho P_x)$, and therefore $\mathcal{E}(\rho_A) = \sum_x P_x\rho_A P_x$.

ii) *If we write the projectors as $P_x = \sum_i |\phi_i^x\rangle\langle\phi_i^x|$, what is the Choi-Jamiołkowski matrix?*

First, note that $P_x = \sum_i |\phi_i^x\rangle\langle\phi_i^x|$ has a sum over the index i, but this sum is not necessarily over each element in the whole space, otherwise $P_x = \mathbb{1}$, and x has only one value (since $\sum_x P_x = \mathbb{1}$). That means we can equivalently write the projectors as: $P_x = \sum_{i\in S_x} |\phi_i\rangle\langle\phi_i|$, where $|\phi_i\rangle$ are the same as $|\phi_i^x\rangle$, but the index x is dropped, since as we'll see in part (iii), these states are orthonormal between x and i. Also, S_x is the set of the full space where we sum over.

From the last exercise, we know that the CJ matrix is given by $\nu = \sum_x |\,P_x\rangle\rangle\,\langle\langle P_x|$. Therefore $\nu_{A'B} = \sum_{x,i,j} |\phi_i^x\rangle_{A'}|\phi_i^x\rangle_B\langle\phi_i^x|_{A'}\langle\phi_i^x|_B = \sum_{x,i\in S_x,j\in S_x} |\phi_i\rangle_{A'}|\phi_i\rangle_B\langle\phi_j|_{A'}\langle\phi_j|_B$ (depending on which notation you use from part (ii)), where the space A' is a copy of the space A (*i.e.* it has the same dimension).

iii) *Give an expression for a purification of the Choi-Jamiołkowski matrix. Note that since the CJ matrix is positive semi-definite and hermitian, you can treat it like an unnormalized density operator.*

First, we show that the states $|\phi_i^x\rangle$ are orthonormal. We know that $P_x^2 = P_x$, and so $P_x^2 = \sum_{x,i,j} |\phi_i^x\rangle\langle\phi_i^x|\phi_j^x\rangle\langle\phi_j^x|$. In order for this to be $P(x)$, $\langle\phi_i^x|\phi_j^x\rangle = \delta_{ij}$. We also know that $\sum_x P_x = \mathbb{1}$, and so squaring this expression gives $\sum_{x,y,i,j} |\phi_i^x\rangle\langle\phi_i^x|\phi_j^y\rangle\langle\phi_j^y| = \mathbb{1}$. If we break this sum up into the sum where $x = y$ and $x \neq y$: $\sum_{x=y,i,j} |\phi_i^x\rangle\langle\phi_i^x| + \sum_{x\neq y,i,j} |\phi_i^x\rangle\langle\phi_i^x|\phi_j^y\rangle\langle\phi_j^y| = \mathbb{1} + \sum_{x\neq y,i,j} |\phi_i^x\rangle\langle\phi_i^x|\phi_j^y\rangle\langle\phi_j^y| = \mathbb{1}$. Therefore we have that $\langle\phi_i^x|\phi_j^y\rangle = \delta_{ij}\delta_{xy}$.

Now we can rewrite ν as $\nu_{A'B} = \sum_x |\psi_x\rangle\langle\psi_x|$, where $|\psi_x\rangle = \sum_i |\phi_i^x\rangle|\phi_i^x\rangle = \sum_{i\in S_x} |\phi_i\rangle\langle\phi_i|$ (depending on which notation from (ii) you use), and we know that the $|\psi_x\rangle$ are orthornormal. Then clearly ν has eigenvectors $|\psi_x\rangle$ with eigenvalues 1.

Now we can write down the purification as: $|\Phi\rangle_{A'BR} = \sum_x |\psi_x\rangle_{A'B}|\varphi_x\rangle_R$, where $|\varphi_x\rangle$ is an orthonormal basis for the space R.

iv) *Apply the inverse of the CJ isomorphism to the purified state in (iii), and show that it is of the form $U\rho_A U^*$, where U is an isometry. The inverse CJ isomorphism is the map that takes a state $\rho_{A'BR}$ as input, and outputs a map $\mathcal{F}$. Specifically:*

$$\mathcal{F}(\rho_A) = \dim(\mathcal{H}_A)\, Tr_{A'}\left(\left(\sum_{i,j} |i\rangle_{A'}\langle j|_A \rho_A |i\rangle_A \langle j|_{A'}\right) \otimes \mathbb{1}_{BR} \cdot \rho_{A'BR}\right),$$

where $\{|i\rangle\}_i$ is an orthonormal basis for A and A' (similarly for $\{|j\rangle\}_j$), and $\rho_{A'BR}$ is the CJ matrix purified.
In place for $\rho_{A'BR}$ we can place the purification of the CJ matrix $|\Phi\rangle_{A'BR}\langle\Phi|$, and normalize it: $\mathrm{Tr}(|\Phi\rangle\langle\Phi|) = \dim(\mathcal{H}_{A'B}) = d^2$ (where $d = \dim(\mathcal{H}_A) = \dim(\mathcal{H}_{A'}) = \dim(\mathcal{H}_B)$), so $|\Phi\rangle \mapsto |\Phi\rangle/d^2$ and rearranging some of the terms:

$$\mathcal{F}(\rho_A) = 1/d \sum_{i,j} \left(((\langle j|_{A'} \otimes \mathbb{1}_{BR})|\Phi\rangle\langle j|_A \rho_A |i\rangle_A \langle\Phi|(|i\rangle_{A'} \otimes \mathbb{1}_{BR})\right).$$

Now let $U = \sqrt{1/d}\sum_j((\langle j|_{A'} \otimes \mathbb{1}_{BR})|\Phi\rangle_{A'BR}\langle j|_A$, so now $\mathcal{F}(\rho_A) = U\rho_A U^*$.
It remains to be shown that U is an isometry. That is

$$\begin{aligned}
U^*U &= 1/d \sum_i |i\rangle_A \langle\Phi|(|i\rangle_{A'} \otimes \mathbb{1}_{BR}) \sum_j ((\langle j|_{A'} \otimes \mathbb{1}_{BR})|\Phi\rangle\langle j|_A \\
&= 1/d \sum_{i,j} |i\rangle_A \langle\Phi|(|i\rangle_{A'}\langle j|_{A'} \otimes \mathbb{1}_{BR})|\Phi\rangle\langle j|_A \\
&= 1/d \sum_{i,j} |i\rangle_A \sum_{x,y} \sum_{l\in S_x, m\in S_y} \langle\phi_l|_B \langle\varphi_x|_R (\langle\phi_l|i\rangle\langle j|\phi_m\rangle \mathbb{1}_{BR})|\phi_m\rangle_B |\varphi_y\rangle_R \langle j|_A \\
&= 1/d \sum_{i,j} |i\rangle_A \langle j|_A \sum_{x,y} \sum_{l\in S_x, m\in S_y} \langle\phi_l|\phi_m\rangle\langle\varphi_x|\varphi_y\rangle(\langle\phi_l|i\rangle\langle j|\phi_m\rangle) \\
&= 1/d \sum_{i,j} |i\rangle_A \langle j|_A \sum_{x,y} \sum_{l\in S_x, m\in S_y} \delta_{l,m}\delta_{x,y}(\langle\phi_l|i\rangle\langle j|\phi_m\rangle) \\
&= 1/d \sum_{i,j} |i\rangle_A \langle j|_A \sum_x \sum_{l\in S_x} \langle\phi_l|i\rangle\langle j|\phi_l\rangle
\end{aligned}$$

We can pick the basis $\{|i\rangle\}_i$ to be same as $\{|\phi_i\rangle\}_i$ and so:

$$\begin{aligned}
U^*U &= 1/d \sum_{i,j} |i\rangle_A \langle j|_A \sum_x \sum_{l\in S_x} \delta_{i,j}\delta_{i,l} \\
&= 1/d \sum_i |i\rangle_A \langle i|_A (\sum_x \sum_{i\in S_x} 1) \\
&= 1/d \sum_i |i\rangle_A \langle i|_A d \\
&= \mathbb{1}_A
\end{aligned}$$

v) *Finally, show that $Tr_R(\mathcal{F}(\rho_A))$ has the same output as the measurement in (i).*
First: $\mathrm{Tr}_R(|\Phi\rangle\langle\Phi|) = \nu_{A'B}$, since it is the purification of $\nu_{A'B}$. Therefore:

$$\begin{aligned}
\mathrm{Tr}_R(\mathcal{F}(\rho_A)) &= 1/d\mathrm{Tr}_{A'}\left(\underbrace{\left(\sum_{i,j} |i\rangle_{A'}\langle j|_A \rho_A |i\rangle_A \langle j|_{A'}\right)}_{\text{Transpose of } \rho} \otimes \mathbb{1}_B \cdot \nu_{A'B}\right) \\
&= \mathrm{Tr}_{A'}\left(\rho^T_{A'} \otimes \mathbb{1}_B/d \cdot \nu_{A'B}\right),
\end{aligned}$$

and this is the CJ representation of the output state, as required.

b) *Give an explicit expression for the map $\mathcal{E}$ for two different measurements on a qubit state described by the POVMs:*

1. $\mathcal{M}_1 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

   CJ Representation:

   Since we know $\nu = \sum_x |P_x\rangle\rangle \langle\langle P_x|$, then $\nu = |00\rangle\rangle \langle\langle 00| + |11\rangle\rangle \langle\langle 11| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   Operator-Sum Representation:

   $\mathcal{E}(\rho) = \langle 0|\rho|0\rangle \otimes |0\rangle\langle 0| + \langle 1|\rho|1\rangle \otimes |1\rangle\langle 1|$.

2. $\mathcal{M}_2 = \{p|0\rangle\langle 0|, p|1\rangle\langle 1|, (1-p)\mathbb{1}_2\}$. *What is the physical interpretation of this POVM?*

   CJ Representation:

   We have to redo part (i) of (a), since now we apply a projector with a certain probability. The post-measurement state is still $\rho_x = P_x \rho P_x^* / \mathrm{Tr}(\rho P_x)$. So now the operator sum notation is $\mathcal{E}(\rho) = \sum_x p(x)\rho_x = \sum_x p_x P_x \rho P_x$, where p_x is the weight associated with each projector (in this case it's p or $1-p$). This makes the CJ matrix $\nu = p_x |P_x\rangle\rangle \langle\langle P_x|$.

   So now $\nu = p|00\rangle\rangle \langle\langle 00| + p|11\rangle\rangle \langle\langle 11| + (1-p)|\mathbb{1}\rangle\rangle \langle\langle \mathbb{1}| = \begin{pmatrix} 1 & 0 & 0 & 1-p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-p & 0 & 0 & 1 \end{pmatrix}$.

   Operator-Sum Representation:

   $\mathcal{E}(\rho) = p\langle 0|\rho|0\rangle \otimes |0\rangle\langle 0| + p\langle 1|\rho|1\rangle \otimes |1\rangle\langle 1| + (1-p)\rho$.

   This measurement can be interpreted as measuring either 0 or 1 with probability p, and otherwise you know the state was left unchanged.

Exercise 8.3 Unambiguous State Discrimination

Suppose you are given one of two states, ρ and σ, with equal probability, and want to distinguish them with a single measurement. We have seen that, unless the states are orthogonal ($\delta(\rho,\sigma) = 1$), it is impossible to always distinguish them with certainty. We also saw that if you wanted to maximize the probability of guessing correctly, the best strategy was to measure the state in the eigenbasis of $\rho - \sigma$: you would be right with probability $\Pr_{\checkmark} = \frac{1}{2}(1 + \delta(\rho,\sigma))$.

Now suppose you have a different goal: you will only make a guess when you are certain of which state you have, so as to never make a mistake. Formally, you will perform a measurement described by a POVM $\{M_\rho, M_\sigma, M_?\}$, such that: (1) if you obtain an outcome corresponding to M_ρ or M_σ, you know for sure that you have ρ or σ, respectively, and (2) if your outcome corresponds to $M_?$ you do not know with certainty which state you have, and you will not risk guessing.

a) *We will consider only pure states $\rho = |\psi\rangle\langle\psi|, \sigma = |\phi\rangle\langle\phi|$. We want to have zero probability of guessing "ψ" when the state measured was ϕ (and vice-versa) . What does this tell us about the form of M_ψ, M_ϕ and $M_?$?*

   In order for the measurement result M_ψ to never occur when the state $|\phi\rangle$ is used, we need that the probability $\langle\phi|M_\psi|\phi\rangle = 0$. This occurs if M_ψ is a linear combination of $|\phi_i^\perp\rangle\langle\phi_i^\perp|$, where $|\phi_i^\perp\rangle$ are normalized states orthogonal to $|\phi\rangle$, i.e. $\langle\phi_i^\perp|\phi\rangle = 0$.

   The same is true for M_ϕ, with $|\psi_i^\perp\rangle\langle\psi_i^\perp|$ respectively.

   Since $M_\psi + M_\phi + M_? = \mathbb{1}$, then we know that $M_? = \mathbb{1} - M_\psi - M_\phi$.

b) *Maximize the probability of making a correct guess, i.e., minimize the probability of obtaining $M_?$. Remember that you can expand one of the states in terms of the other and a particular vector orthogonal to it denoted by index k or l, for instance*

$$|\phi\rangle = a|\psi\rangle + b|\psi_k^\perp\rangle, \quad |\psi\rangle = a^*|\phi\rangle - b|\phi_l^\perp\rangle, \qquad a = \langle\psi|\phi\rangle, \quad |a|^2 + |b|^2 = 1. \tag{2}$$

From part (a) we know that $M_\psi = \sum_i c_{i,\psi} |\phi_i^\perp\rangle\langle\phi_i^\perp|$, $M_\phi = \sum_j c_{j,\phi} |\psi_j^\perp\rangle\langle\psi_j^\perp|$ and $M_? = \mathbb{1} - M_\psi - M_\phi$, where $c_{i,\psi}$, $c_{j,\phi}$ are positive constants smaller or equal to 1.

The probability of succeeding is given by

$$p_\checkmark = \frac{1}{2}(\langle\phi|M_\phi|\phi\rangle + \langle\psi|M_\psi|\psi\rangle) = \sum_{i,j} \frac{1}{2}(c_{i,\phi}|\langle\phi|\psi_i^\perp\rangle|^2 + c_{j,\psi}|\langle\psi|\phi_j^\perp\rangle|^2). \tag{3}$$

If we replace $|\phi\rangle$ and $|\psi\rangle$ with their representation as in (2) we get

$$p_\checkmark = \frac{1}{2}(c_{k,\phi}|\langle\psi_k|b|\psi_k^\perp\rangle|^2 + c_{l,\psi}|(\langle\phi_l^\perp|(-b)|\phi_l^\perp\rangle)|^2) = \frac{1}{2}(c_{k,\phi} + c_{l,\psi})|b|^2. \tag{4}$$

The most we can make $c_{k,\phi}$ and $c_{l,\psi}$ is 1 and so we have $p_\checkmark = 1 - |\langle\psi|\phi\rangle|^2$ since $|b|^2 = 1 - |a|^2 = 1 - |\langle\psi|\phi\rangle|^2$.

c) *What happens if ψ and ϕ are given with probability q and $1 - q$?*

Redoing the calculation of part (b) we get

$$p_\checkmark = ((1-q)\langle\phi|M_\phi|\phi\rangle + q\langle\psi|M_\psi|\psi\rangle) = ((1-q)c_{k,\phi} + qc_{l,\psi})|b|^2. \tag{5}$$

The most we can make $c_{k,\phi}$ and $c_{l,\psi}$ is 1, and since $1 - q$ and q are positive, we get the same result as in (b), i.e. $p_\checkmark = 1 - |\langle\psi|\phi\rangle|^2$, and the measurement operators are the same, $M_\phi, M_\psi, M_?$.