ETH
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Quantum Information Theory
Solutions 7

HS 12
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Exercise 7.1 Distinguishing Channels

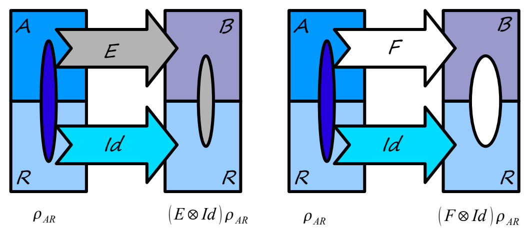
We have seen that TPCPMs may be used to define channels. Now let's see how to quantify similarity between two channels. Consider two TPCPMs $\mathcal{E}, \mathcal{F} : End(\mathcal{H}_A) \mapsto End(\mathcal{H}_B)$.
A naive approach is to send the same state through each of the channels and see how similar the output states are,

$$d(\mathcal{E}, \mathcal{F}) = \max_{\rho_A} \delta(\mathcal{E}(\rho_A), \mathcal{F}(\rho_A)), \tag{1}$$

where $\delta(\rho, \sigma)$ is the trace distance between states.
However, we may want to consider that ρ_A may be entangled with some other system, and therefore a channel that acts locally may produce global changes on the total state (for instance break the entanglement). The stabilized distance takes that into account:

$$d^{\diamond}(\mathcal{E}, \mathcal{F}) = \max_{\rho_{AR}} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_{AR}), \mathcal{F} \otimes \mathcal{I}(\rho_{AR})), \tag{2}$$

where $\mathcal{I}$ is the identity map.



a) *Show that in general $d(\mathcal{E}, \mathcal{F}) \leq d^{\diamond}(\mathcal{E}, \mathcal{F})$.*

First start with the definition of $d^{\diamond}(\mathcal{E}, \mathcal{F})$:

$$\begin{aligned}
\max_{\rho_{AR}} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_{AR}), \mathcal{F} \otimes \mathcal{I}(\rho_{AR})) &\geq \max_{\rho_A \otimes \rho_R} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_A \otimes \rho_R), \mathcal{F} \otimes \mathcal{I}(\rho_A \otimes \rho_R)) && (3)\\
&= \max_{\rho_A \otimes \rho_R} \frac{1}{2} \mathrm{Tr}_{AR} \left| \mathcal{E} \otimes \mathcal{I}(\rho_A \otimes \rho_R) - \mathcal{F} \otimes \mathcal{I}(\rho_A \otimes \rho_R) \right| && (4)\\
&= \max_{\rho_A \otimes \rho_R} \frac{1}{2} \mathrm{Tr}_{AR} \left| (\mathcal{E}(\rho_A) - \mathcal{F}(\rho_A)) \otimes \rho_R) \right| && (5)\\
&= \max_{\rho_A} \frac{1}{2} \mathrm{Tr}_A \left| (\mathcal{E}(\rho_A) - \mathcal{F}(\rho_A)) \right| && (6)\\
&= \delta(\mathcal{E}, \mathcal{F}) && (7)
\end{aligned}$$

b) *Compute and compare $d(\mathcal{E}, \mathcal{F})$ and $d^{\diamond}(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ act on ρ as $\mathcal{E}_{\mathcal{A}}(\rho_A) = \mathcal{I}(\rho_A)$ and $\mathcal{F}_{\mathcal{A}}(\rho_A) = \frac{\mathbb{I}_A}{d_A}$.*

First we calculate $d(\mathcal{E}, \mathcal{F})$:

$$\begin{aligned}
d(\mathcal{E}, \mathcal{F}) &= \max_{\rho_A} \delta\left(\rho_A, \frac{\mathbb{1}_A}{d_A}\right) && (8)\\
&= \max_{\rho_A} \frac{1}{2} \mathrm{Tr} \left| \frac{\mathbb{1}_A}{d_A} - \rho_A \right|. && (9)
\end{aligned}$$

Note that $1/2\mathrm{Tr}|\mathbb{1}/d_A - \rho|$ is the distance between ρ and the center of the Bloch sphere when $d_A = 2$. From Claim 4.4.12 in the script, the maximum occurs when ρ_A is a pure state. Therefore,

$$d(\mathcal{E}, \mathcal{F}) = \frac{1}{2}\left(1 - \frac{1}{d_A} + \frac{d_A - 1}{d_A}\right) = \frac{d_A - 1}{d_A}.$$

As for the diamond distance, we have

$$
\begin{aligned}
d^{\diamond}(\mathcal{E},\mathcal{F}) &= \max_{\rho_{AR}} \delta(\mathcal{I}\otimes\mathcal{I}(\rho_{AR}), \mathcal{F}\otimes\mathcal{I}(\rho_{AR})) \\
&= \max_{\rho_{AR}} \frac{1}{2}\mathrm{Tr}\left|\rho_{AR} - \frac{\mathbb{1}_A}{d_A}\otimes\rho_R\right|
\end{aligned}
$$

First, it can be shown that a pure state will maximize this expression. Using the Schmidt decomposition $\rho_{AR} = \sum_{ij}\gamma_i\gamma_j|ii\rangle_{AR}\langle jj|$, and writing $d = d_A$ we get

$$
d^{\diamond}(\mathcal{E},\mathcal{F}) = \max_{\rho_{AR}} \frac{1}{2}\mathrm{Tr}\left|\sum_{ij}\gamma_i\gamma_j|ii\rangle_{AR}\langle jj| - \frac{\mathbb{1}_A}{d}\otimes\sum_k\gamma_k^2|k\rangle_R\langle k|\right|
$$

$$
= \max_{\rho_{AR}} \frac{1}{2}\mathrm{Tr}\left|\begin{pmatrix}
\gamma_1^2 \\
& 0 \\
& & \ddots \\
& & & 0 \\
& & & & 0 \\
& & & & & \gamma_2^2 \\
& & & & & & \ddots \\
& & & & & & & 0 \\
& & & & & & & & \ddots \\
& & & & & & & & & 0 \\
& & & & & & & & & & \ddots \\
& & & & & & & & & & & 0 \\
& & & & & & & & & & & & \gamma_d^2
\end{pmatrix} - \frac{1}{d}\begin{pmatrix} 1 \\ & 1 \\ & & \ddots \\ & & & 1\end{pmatrix}\otimes\begin{pmatrix}\gamma_1^2 \\ & \gamma_2^2 \\ & & \ddots \\ & & & \gamma_d^2\end{pmatrix}\right|
$$

$$
= \max_{\rho_{AR}} \frac{1}{2d}\mathrm{Tr}\left|\begin{pmatrix}
\gamma_1^2(d-1) \\
& -\gamma_2^2 \\
& & \ddots \\
& & & -\gamma_d^2 \\
& & & & -\gamma_1^2 \\
& & & & & \gamma_2^2(d-1) \\
& & & & & & \ddots \\
& & & & & & & -\gamma_d^2 \\
& & & & & & & & \ddots \\
& & & & & & & & & -\gamma_1^2 \\
& & & & & & & & & & \ddots \\
& & & & & & & & & & & -\gamma_{d-1}^2 \\
& & & & & & & & & & & & \gamma_d^2(d-1)
\end{pmatrix}\right|
$$

$$
= \max_{\rho_{AR}} \frac{1}{2d}\left(\sum_i\gamma_i^2(d-1) + (d-1)\sum_i\gamma_i^2\right) = \frac{d-1}{d}
$$

Therefore, the stabilized distance and the trace distance are the same in this case.

Exercise 7.2 Bell-type Experiment

Consider a 2-qubit Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A\otimes\mathcal{H}_B$ with basis $\{|00\rangle, |01\rangle|10\rangle, |11\rangle\}$ in the Bell-state

$$
|\psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right). \tag{10}
$$

Two parties, called Alice and Bob, get half of the state $|\psi^+\rangle$ so that Alice has system A, and Bob has system B. Alice then performs a measurement $\mathcal{M}_A^\alpha := \{|\alpha\rangle\langle\alpha|, |\alpha\rangle^\perp\langle\alpha|^\perp\}$, with $|\alpha\rangle := \cos(\frac{\alpha}{2})|0\rangle + \sin(\frac{\alpha}{2})|1\rangle$, on her part of the system.

a) *Find the description Bob would give to his partial state on B after he knows that Alice performed the measurement M_A^α on A. What description would Alice give to ρ_B given that she knows what measurement outcome she received?*

Let $P_\alpha = |\alpha\rangle\langle\alpha| \otimes \mathbb{1}_B$ and $P_{\alpha^\perp} = |\alpha^\perp\rangle\langle\alpha^\perp| \otimes \mathbb{1}_B$.

We have

$$|\alpha\rangle = \cos\frac{\alpha}{2}|0\rangle + \sin\frac{\alpha}{2}|1\rangle \qquad |\alpha\rangle\langle\alpha| = \begin{pmatrix} \cos^2\frac{\alpha}{2} & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

$$P_\alpha = \begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 \\ 0 & \cos^2\frac{\alpha}{2} & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \sin^2\frac{\alpha}{2} & 0 \\ 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

$$|\alpha^\perp\rangle = \cos\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)|0\rangle + \sin\left(\frac{\alpha}{2}+\frac{\pi}{2}\right)|1\rangle \qquad |\alpha^\perp\rangle\langle\alpha^\perp| = \begin{pmatrix} \cos^2\frac{\alpha}{2} & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}$$

$$= -\sin\frac{\alpha}{2}|0\rangle + \cos\frac{\alpha}{2}|1\rangle$$

$$P_{\alpha^\perp} = \begin{pmatrix} \sin^2\frac{\alpha}{2} & 0 & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 \\ 0 & \sin^2\frac{\alpha}{2} & 0 & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \cos^2\frac{\alpha}{2} & 0 \\ 0 & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & \cos^2\frac{\alpha}{2} \end{pmatrix},$$

and the state shared by Alice and Bob is

$$|\Psi^+\rangle\langle\Psi^+| = \frac{|00\rangle\langle00| + |00\rangle\langle11| + |11\rangle\langle00| + |11\rangle\langle11|}{2} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

The probability that Alice obtains outcome α is

$$\Pr_A(\alpha) = \mathrm{Tr}\Big[P_\alpha|\psi^+\rangle\langle\psi^+|\Big] = \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 & 0 & \cos^2\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \sin^2\frac{\alpha}{2} & 0 & 0 & \sin^2\frac{\alpha}{2} \end{pmatrix}\right] = \frac{1}{2}$$

and the reduced state on Bob's side after the measurement, from the point of view of Alice (who knows that the outcome of her measurement was α) is

$$\rho_{B|A=\alpha} = \mathrm{Tr}_A\left(\frac{P_\alpha|\psi^+\rangle\langle\psi^+|}{\Pr_A(\alpha)}\right) = \mathrm{Tr}_A\begin{pmatrix} \cos^2\frac{\alpha}{2} & 0 & 0 & \cos^2\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & 0 & 0 & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \sin^2\frac{\alpha}{2} & 0 & 0 & \sin^2\frac{\alpha}{2} \end{pmatrix} = \begin{pmatrix} \cos^2\frac{\alpha}{2} & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix}.$$

Similarly, for the the other outcome, $\alpha^\perp$, we have

$$\Pr_A(\alpha^\perp) = \frac{1}{2} \qquad \rho_{B|A=\alpha^\perp} = \begin{pmatrix} \sin^2\frac{\alpha}{2} & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \cos^2\frac{\alpha}{2} \end{pmatrix}.$$

If Bob does not know that Alice performed a measurement, he sees his state as $\rho_B = \mathrm{Tr}_A|\phi^+\rangle\langle\phi^+| = \frac{1}{2}\mathbb{1}_B$. Imagine now that Bob knows that Alice performed this measurement but does not know the outcome. All he know is that he has state $\rho_{B|A=\alpha}$ if she got α and state $\rho_{B|A=\alpha^\perp}$ if she got $\alpha^\perp$,

$$\begin{aligned} \rho_{B|A=?} &= \Pr_A(\alpha)\rho_{B|A=\alpha} + \Pr_A(\alpha^\perp)\rho_{B|A=\alpha^\perp} \\ &= \frac{1}{2}\begin{pmatrix} \cos^2\frac{\alpha}{2} & \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ \cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \sin^2\frac{\alpha}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sin^2\frac{\alpha}{2} & -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} & \cos^2\frac{\alpha}{2} \end{pmatrix} = \frac{\mathbb{1}_B}{2}. \end{aligned}$$

From Bob's viewpoint, it is the same whether Alice does not measure her qubit or measures it but does not tell him the outcome.

b) *If Bob does the measurement $\mathcal{M}_B^0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on B, what is the probability distribution for his outcomes, Pr_B? How would Alice describe his probability distribution, $Pr_{B|A}$?*

We can read the probability distribution of $\mathcal{M}_B^0$ conditioned on A directly from our results in part a),

$$\Pr_{B|A=\alpha}(x) = \begin{cases} \cos^2 \frac{\alpha}{2}, & \text{for } x = 0 \\ \sin^2 \frac{\alpha}{2}, & \text{for } x = 1 \end{cases}$$

For $A = \alpha^\perp$ the probabilities are interchanged. P_B (not conditioned on A) is just the uniform distribution over $0, 1$.

c) *Explain how this subjective assignment of the scenarios at B does not contradict with the actual measurement outcomes Bob will get after doing the measurement $\mathcal{M}_B^0$..*

The qubit of system B is correlated to another system (A) and what we are looking at are the states (and probability distributions) conditioned / not conditioned on an event on that system (measurement on A) that is itself random. More detailed analysis in the tips.

d) *Alice and Bob can choose two different bases each: $\alpha = 0, \frac{\pi}{2}$ for Alice (labelled bases 0 and 3) and $\alpha = \frac{\pi}{4}, \frac{3\pi}{4}$ for Bob (1 and 3).*

The joint probabilities $P_{XY|ab}(x, y)$ of them obtaining outcomes x and y when they measure $A = a$ and $B = b$ are given by

Alice / Bob		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2} - \varepsilon$	ε	$\frac{1}{2} - \varepsilon$	ε
	−	ε	$\frac{1}{2} - \varepsilon$	ε	$\frac{1}{2} - \varepsilon$
B=3	+	ε	$\frac{1}{2} - \varepsilon$	$\frac{1}{2} - \varepsilon$	ε
	−	$\frac{1}{2} - \varepsilon$	ε	ε	$\frac{1}{2} - \varepsilon$

with $\varepsilon = \frac{1}{2} \sin^2(\pi/8) \approx 0.07$.

Compute

$$I_N(P_{XY|AB}) = P(X = Y|A = 0, B = 3) + \sum_{|a-b|=1} P(X \neq Y|A = a, B = b).$$

$$I_N(P_{XY|AB}) = \sum_{k=+,-} \left[P_{XY|A=0,B=3}(k, k) + \sum_{|a-b|=1} P_{XY|A=a,B=b}(k, \bar{k}) \right]$$
$$= \sum \text{red terms in the table} = 8\varepsilon$$

e) *Now we introduce a PR-box, which is a joint probability distribution that cannot be created by measurements on a quantum state:*

Alice / Bob		A =0 +	A =0 −	A =2 +	A =2 −
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Show that the PR-box

(i) *is non-signalling: $P(X|a, b_1) = P(X|a, b_2), \forall a$;*

$$P_{X|A=a,B=1}(x) = P_{X|A=a,B=3}(x), \ \forall a, x \Leftrightarrow$$
$$\Leftrightarrow \sum_y P_{XY|A=a,B=1}(x, y) = \sum_y P_{X|A=a,B=3}(x, y), \ \forall a, x \Leftrightarrow$$
$$\Leftrightarrow \sum \text{red terms} = \sum \text{orange terms}, \ \forall \text{ columns} \Leftrightarrow$$
$$\Leftrightarrow \frac{1}{2} = \frac{1}{2} \ \checkmark$$

The other non-signalling condition, $P(Y|a_1, b) = P(Y|a_2, b), \forall b$, can be checked similarly, summing over rows instead of columns.

(ii) *is non-local:* $P_{XY|ab} \neq P_{X|a} P_{Y|b}$*;*
Any local distribution is a convex combination of deterministic local distributions,

$$v$$

There are only 16 non-signalling deterministic local distributions:

$$\begin{array}{|cc|cc|} \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array}, \quad \begin{array}{|cc|cc|} \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline \end{array}, \quad \begin{array}{|cc|cc|} \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline \end{array}, \quad \begin{array}{|cc|cc|} \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hline \end{array}, \quad \text{etc.}$$

The However, if we try to decompose the PR-box in this way we obtain:

$$\begin{array}{|cc|cc|} \hline \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \hline \end{array} = \frac{1}{2} \begin{array}{|cc|cc|} \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array} + \frac{1}{2} \begin{array}{|cc|cc|} \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array}$$

$$P_{XY|A=a,B=b}(x,y) \neq P_{X|A=a}(x)\ P_{Y|B=b}(y),\ \forall a,b,x,y \Leftrightarrow$$
$$\Leftrightarrow P_{XY|A=a,B=b}(x,y) \neq \left[\sum_{y',b'} P_{XY|A=a,B=b'}(x,y')\right] \left[\sum_{x',a'} P_{XY|A=a',B=b}(x',y)\right],\ \forall a,b,x,y \Leftrightarrow$$
$$\Leftrightarrow \{\text{table cell}\} \neq \left[\sum \{\text{column of the cell}\}\right] \left[\sum \text{row of the cell}\right],\ \forall \text{ cells} \Leftrightarrow$$
$$\Leftrightarrow 0 \text{ or } \frac{1}{2} \neq \left[\frac{1}{2} + \frac{1}{2}\right] \left[\frac{1}{2} + \frac{1}{2}\right] = 1\ \checkmark$$

(iii) *yields* $I_N(P_{XY|AB}) = 0$*.*

Alice		A =0		A =2	
Bob		+	−	+	−
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

$$I_N(P_{XY|AB}) = \sum \text{red terms} = 0$$

f) Consider now that Alice and Bob get their qubits and measurement devices from Eve. Eve will try to trick them into thinking that they share a singlet and perform quantum measurements. In fact, she will give them a device that allows her to guess the results of their "measurements" with some probability.

Eve is a post-quantum adversary, limited only by non-signaling. She will give them:

- *with probability* $1 - p$ *a PR-box;*
- *with probability* $p/4$*, one of four deterministic boxes, that always outcome* $++$*,* $+-$*,* $-+$ *and* $--$ *respectively.*

Find p so that the final joint probability distribution equals the one of the singlet state. What is the probability that Eve can guess the outcomes of their measurements?

If we look at the first entry in the table (top left), it is straightforward to see that we need

$$(1-p) * \frac{1}{2} + p * \frac{1}{4} = \frac{1}{2} - \epsilon, \tag{11}$$

which implies that $p = 4\epsilon$. One can easily verify that this result also works for the other entries in the table.