Exercise 4.1 Bloch sphere

We will see that density operators of two-level systems (qubits) can always be expressed as

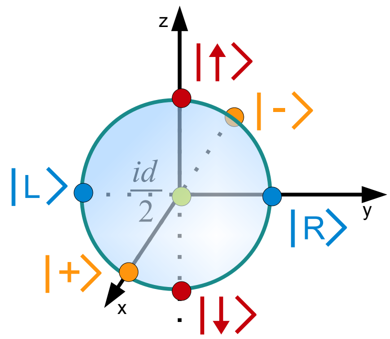
$$\rho = \frac{1}{2}(\mathbb{1} + \vec{r} \cdot \vec{\sigma}) \tag{1}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\vec{r} = (r_x, r_y, r_z), |\vec{r}| \leq 1$ is the so-called Bloch vector, that gives us the position of a point in a unit ball. The surface of that ball is usually known as the Bloch sphere.

a) *Using Eq. 1 :*

1) *Find and draw in the ball the Bloch vectors of a fully mixed state and the pure states that form three bases, $\{|\uparrow\rangle, |\downarrow\rangle\}$, $\{|+\rangle, |-\rangle\}$ and $\{|\circlearrowleft\rangle, |\circlearrowright\rangle\}$.*

state	**density matrix**	**Bloch vector**	**in the figure**
$\frac{\mathbb{1}}{2}$	$\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$(0,0,0)$	green
$\|0\rangle$	$\frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$	$(0,0,1)$	red
$\|1\rangle$	$\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$	$(0,0,-1)$	red
$\|+\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	$(1,0,0)$	yellow
$\|-\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$	$(-1,0,0)$	yellow
$\|\circlearrowright\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$	$(0,1,0)$	blue: $\|R\rangle$
$\|\circlearrowleft\rangle$	$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$	$(0,-1,0)$	blue: $\|L\rangle$



2) *Find and diagonalise the states represented by Bloch vectors $\vec{r}_1 = (\frac{1}{2}, 0, 0)$ and $\vec{r}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.*

We have

$$\begin{aligned}
\rho_1 &= \frac{1}{2}\left[\mathbb{1} + \left(\frac{1}{2}, 0, 0\right) \cdot (\sigma_x, \sigma_y, \sigma_z)\right] \\
&= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \\
&= \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad \Rightarrow \qquad \text{Eigenvalues: } \left\{\frac{1}{4}, \frac{3}{4}\right\},
\end{aligned}$$

$$\rho_2 = \frac{1}{2}\left[\mathbb{1} + \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot(\sigma_x,\sigma_y,\sigma_z)\right]$$
$$= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$
$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{2}+1 & 1 \\ 1 & \sqrt{2}-1 \end{pmatrix} \quad \Rightarrow \quad \text{Eigenvalues: } \{0,1\}\,.$$

The first Bloch vector lies inside the ball ($|\vec{r}_1 = \frac{1}{4}|$), and the state that it represents is mixed. The Bloch vector of the second state is on the surface of the sphere, and that state is pure.

b) *Show that the operator ρ defined in Eq. 1 is a valid density operator for any vector $\vec{r}$ with $|\vec{r}| \leq 1$ by proving it fulfils the following properties:*

1) *Hermiticity: $\rho = \rho^\dagger$.*
   All Pauli matrices are Hermitian and the vector $\vec{r}$ is real, so the result comes from direct application of Eq. 1.
2) *Positivity: $\rho \geq 0$.*
   The general form of a state given by Eq. 1 is
   $$\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix} \quad \Rightarrow \quad \text{Eigenvalues: } \left\{\frac{1-|\vec{r}|}{2}, \frac{1+|\vec{r}|}{2}\right\}. \tag{2}$$
   Since $0 \leq |\vec{r}| \leq 1$, the eigenvalues are non negative.
3) *Normalisation: $Tr(\rho) = 1$.*
   From Eq. 2 we have that
   $$\mathrm{Tr}(\rho) = \frac{1-|\vec{r}|}{2} + \frac{1+|\vec{r}|}{2} = 1.$$

c) *Now do the converse: show that any two-level density operator may be written as Eq. 1.*

One can always expand an operator A in an orthonormal basis $\{e_i\}_i$ as
$$A = \sum_i (A, e_i) e_i,$$
where the inner product (A,B) is defined as $\mathrm{Tr}(A^* B)$.

The three Pauli matrices and the identity form a basis for 2×2 matrices, $\mathcal{B}$. However, this basis is not normalized. A normalised basis would be
$$\mathcal{B}' = \left\{\frac{\sigma_x}{\sqrt{2}}, \frac{\sigma_y}{\sqrt{2}}, \frac{\sigma_z}{\sqrt{2}}, \frac{\mathbb{1}}{\sqrt{2}}\right\}. \tag{3}$$

We can expand any 2×2 matrix in this basis, and in particular any two-level density operator:
$$\rho = \mathrm{Tr}(\rho\mathbb{1})\,\frac{\mathbb{1}}{2} + \sum_i \mathrm{Tr}(\rho\sigma_i)\,\frac{\sigma_i}{2} \tag{4}$$
$$= \frac{\mathbb{1}}{2} + \frac{1}{2}\Big(\mathrm{Tr}(\rho\sigma_x), \mathrm{Tr}(\rho\sigma_y), \mathrm{Tr}(\rho\sigma_z)\Big)\cdot\Big(\sigma_x,\sigma_y,\sigma_z\Big) \tag{5}$$
$$= \frac{\mathbb{1}}{2}\,(r_x, r_y, r_z)\cdot(\sigma_x,\sigma_y,\sigma_z) \tag{6}$$

Where we used the property of density operators $\mathrm{Tr}(\rho) = 1$. To obtain the bound $|\vec{r}| \leq 1$ we use the fact that for any density operator $\mathrm{Tr}(\rho^2) \leq 1$ (because all eigenvalues $\lambda_j \leq 1$ and $\sum \lambda_j = 1$) and get

$$\begin{aligned}
1 &\geq \mathrm{Tr}(\rho^2) \\
&= \mathrm{Tr}\Big(\Big[\frac{\mathbb{1}}{2} + \sum_i r_i \frac{\sigma_i}{2}\Big] \Big[\frac{\mathbb{1}}{2} + \sum_i r_i \frac{\sigma_i}{2}\Big] \Big) \\
&= \frac{1}{4}\mathrm{Tr}\Big(\Big[1 + \sum_i {r_i}^2\Big] \mathbb{1} \Big) \qquad \text{(because } \mathcal{B}' \text{ is an orthonormal basis)} \\
&= \frac{1}{2}\Big(1 + \sum_i {r_i}^2\Big) \\
1 &\geq \sum_i {r_i}^2.
\end{aligned}$$

d) *Check that the surface of the ball — the Bloch sphere — is formed by all the pure states.*

For pure states, $\mathrm{Tr}(\rho^2) = 1$ and we can replace all "$\geq$" with "$=$" above, obtaining $|\vec{r}| = 1$.

e) *Discuss the analog of the Bloch sphere in higher dimensions. What can be said? For instance, where are the pure states?*

`http://en.wikipedia.org/wiki/Bloch_sphere#A_generalization_for_pure_states` :)

Exercise 4.2 Partial trace

Given a density matrix ρ_{AB} on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_A = Tr_B \rho_{AB}$,

a) *Show that ρ_A is a valid density operator by proving it is:*

1) *Hermitian: $\rho_A = \rho_A^\dagger$.*
Remember that ρ_{AB} can always be written as

$$\rho_{AB} = \sum_{i,j,k,l} c_{ij;kl}\ |i\rangle\langle k|_A \otimes |j\rangle\langle l|_B,$$

where $c_{ij;kl} = c^\dagger_{kl;ij}$ is hermitian.
The reduced density operator ρ_A is then given by

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \sum_{i,k}\sum_m c_{im;km}|i\rangle\langle k|_A$$

as can easily be verified. Hermiticity of ρ_A follows from

$$\rho_A^\dagger = \sum_{i,k}\sum_m c^\dagger_{im;km}\,(|i\rangle\langle k|_A)^\dagger = \sum_{i,k}\sum_m c_{km;im}|k\rangle\langle i|_A = \rho_A.$$

2) *Positive: $\rho_A \geq 0$.*
Since $\rho_{AB} \geq 0$ is positive, its scalar product with any pure state is positive. Let $|\Psi_m\rangle_{AB} = |\psi\rangle_A \otimes |m\rangle_B$ be a state in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $|\psi\rangle_A$ an arbitrary pure state in $\mathcal{H}_A$:

$$\begin{aligned}
0 &\leq \sum_m \langle \Psi_m|\rho_{AB}|\Psi_m\rangle \\
&= \sum_m \langle\psi|_A \otimes \langle m|_B \rho_{AB} |\psi\rangle_A \otimes |m\rangle_B \\
&= \sum_m \sum_{i,j,k,l} c_{ij;kl}\langle\psi|i\rangle\langle k|\psi\rangle_A \langle m|j\rangle\langle l|m\rangle_B \\
&= \sum_{i,k}\sum_m c_{im;km}\langle\psi|i\rangle\langle k|\psi\rangle_A \\
&= \langle\psi|\rho_A|\psi\rangle
\end{aligned}$$

Because this is true for any $|\psi\rangle$, it follows that ρ_A is positive.

3) *Normalised: Tr(ρ_A) = 1.*

$$\begin{aligned}\mathrm{Tr}(\rho_A) &= \sum_{i,j}\sum_{m,n} c_{im;km}\langle n|i\rangle\langle k|n\rangle \\ &= \sum_{m,n} c_{nm;nm} = \mathrm{Tr}(\rho_{AB}) = 1.\end{aligned}$$

b) *Calculate the reduced density matrix of system A in the Bell state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.*

The reduced state is mixed, even though $|\Psi\rangle$ is pure:

$$\begin{aligned}\rho_{AB} = |\Psi\rangle\langle\Psi| &= \frac{1}{2}\Big(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\Big) \\ \mathrm{Tr}_B(\rho_{AB}) &= \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big) = \frac{1}{2}\mathbb{1}_A.\end{aligned}$$

c) *Consider a classical probability distribution P_{XY} with marginals P_X and P_Y.*

1) *Calculate the marginal distribution P_X for*

$$P_{XY}(x,y) = \begin{cases} 0.5 & \text{for } (x,y) = (0,0), \\ 0.5 & \text{for } (x,y) = (1,1), \\ 0 & \text{else,} \end{cases} \tag{7}$$

with alphabets $\mathcal{X}, \mathcal{Y} = \{0,1\}$.
Using $P_X(x) = \sum_y P_{XY}(x,y)$, we obtain

$$P_X(0) = 0.5, \quad P_X(1) = 0.5.$$

2) *How can we represent P_{XY} in form of a quantum state?*
A probability distribution $P_Z = \{P_Z(z)\}_z$ may be represented by a state

$$\rho_Z = \sum_z P_Z(z)|z\rangle\langle z|, \tag{8}$$

for a basis $\{|z\rangle\}_z$ of a Hilbert space $\mathcal{H}_Z$. In this case we can create a two-qubit system with composed Hilbert space $\mathcal{H}_X\mathcal{H}_Y$ in state

$$\rho_{XY}\frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big).$$

3) *Calculate the partial trace of P_{XY} in its quantum representation.*
The reduces state of qubit X is

$$\rho_X = \frac{1}{2}\big(|0\rangle\langle 0| + |1\rangle\langle 1|\big).$$

Notice that the reduced states of this classical state and the Bell state are the same, the state of the global state is very different — in particular, the latter is a pure state that can be very useful in quantum communication and cryptography.

d) *Can you think of an experiment to distinguish the bipartite states of parts b) and c)?*

One could for instance measure the two states in the Bell basis,

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad |\psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad |\psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

The Bell state we analised corresponds to the first state of this basis, $|\Psi\rangle = |\psi_1\rangle$, and a measurement in the Bell basis would always have the same outcome. For the classical state, however, $\rho_{XY} = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)$, so with probability $\frac{1}{2}$ a measurement in this basis will output $|\psi_2\rangle$, and we will know we had the classical state.