Quantum Information Theory
Series 10

Definitions: von Neumann entropy

In this series we will derive some useful properties of the von Neumann entropy. We will also look at quantum mutual information. Before we start, here are a few definitions.
The von Neumann entropy of a density operator $\rho \in \mathcal{S}(\mathcal{H}_A)$ is defined as

$$H(A)_\rho := -\mathrm{Tr}\big(\rho \log \rho\big) = -\sum_i \lambda_i \log \lambda_i, \tag{1}$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ.
Given a composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we write the entropy of the reduced state of a subsystem, such as $\mathcal{H}_A \otimes \mathcal{H}_B$ as $H(AB)_\rho$, where $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$. When the state ρ is obvious from the context we do not write it as the subscript.
The *conditional* von Neumann entropy is defined as

$$H(A|B)_\rho := H(AB)_\rho - H(B)_\rho. \tag{2}$$

In the Alice-and-Bob picture this quantifies the uncertainty that Bob (who holds the B part of the quantum state ρ_{AB}) has about Alice's state ρ_A.
The strong sub-additivity property of the von Neumann entropy is very useful. It applies to a tripartite composite system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$,

$$H(A|BC)_\rho \le H(A|B)_\rho. \tag{3}$$

Exercise 10.1 Upper bound on von Neumann entropy

Given a state $\rho \in \mathcal{S}(\mathcal{H}_A)$, show that

$$H(A)_\rho \le \log |\mathcal{H}_A|. \tag{4}$$

There are several ways to do this. In this exercise, you should do it as follows.
Consider the state $\bar{\rho} = \int U \rho U^\dagger dU$, where the integral is over all unitaries $U \in \mathcal{U}(\mathcal{H})$ and dU is the Haar measure. Find $\bar{\rho}$ and show (4) using concavity:

$$H(A)_\rho \ge \sum_z p_z \, H(A|Z = z). \tag{5}$$

The Haar measure satisfies $d(UV) = d(VU) = dU$, where $V \in \mathcal{U}(\mathcal{H})$ is a unitary.

Exercise 10.2 Quantum mutual information

One way of quantifying correlations between two systems A and B is through their *mutual information*, defined as

$$\begin{aligned} I(A:B)_\rho &:= H(A)_\rho + H(B)_\rho - H(AB)_\rho & (6)\\ &= H(A)_\rho - H(A|B)_\rho. & (7)\end{aligned}$$

We can also define a conditional version of the mutual information between A and B as

$$\begin{aligned} I(A:B|C)_\rho &:= H(A|C)_\rho + H(B|C)_\rho - H(AB|C)_\rho & (8)\\ &= H(A|C)_\rho - H(A|BC)_\rho. & (9)\end{aligned}$$

a) Consider two qubits A and B in joint state ρ_{AB}.

1. Prove that the mutual information of the Bell state $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is maximal. This is why we say Bell states are *maximally entangled*.
2. Show that $I(A:B) \le 1$ for classically correlated states, $\rho_{AB} = p|0\rangle\langle 0|_A \otimes \sigma_B^0 + (1-p)|1\rangle\langle 1|_A \otimes \sigma_B^1$ (where $0 \le p \le 1$).

b) Consider the so-called *cat state* of four qubits, $A \otimes B \otimes C \otimes D$, that is defined as

$\Big|$  $\Big\rangle = \frac{1}{\sqrt{2}} \left(|0000\rangle + |1111\rangle \right).$ (10)

Show that the mutual information between A and B changes with the knowledge of the remaining qubits,

1. $I(A:B) = 1$.
2. $I(A:B|C) = 0$.
3. $I(A:B|CD) = 1$.

How do you interpret these results for the mutual information of the cat state?

Exercise 10.3 Information measures bonanza

Take a system A in state ρ. Non-conditional quantum min- and max-entropies are given by

$$H_{\min}(A)_\rho = -\log \max_{\lambda \in \mathrm{EV}(\rho)} \lambda, \qquad H_{\max}(A)_\rho = \log \operatorname{rank}(\rho).$$

For instance, if ρ_A has eigenvalues $\mathrm{EV}(\rho_A) = \{0.6, 0.2, 0.2, 0\}$, we have $H_{\min}(A)_\rho = -\log 0.6$ and $H_{\max}(A)_\rho = \log 3$.

a) Show that if $\mathrm{EV}(\rho) \prec \mathrm{EV}(\tau)$, then the entropy of ρ is larger than or equal to the entropy of τ, for the von Neumann, min- and max-entropies.

b) Show that if the bipartite state $|\psi\rangle_{AB}$ can be transformed into $|\phi\rangle$ via LOCC (without catalysts), then $I(A:B)_\psi \geq I(A:B)_\phi$.