Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Quantum Information Theory
Series 7

HS 12
Prof. R. Renner

Exercise 7.1 Distinguishing channels

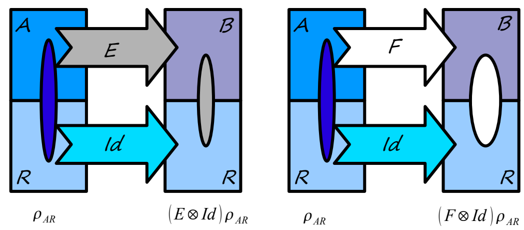
We have seen that CPTPMs may be used to define channels. Now let's see how to quantify the difference between two channels. Consider two CPTPMs $\mathcal{E}, \mathcal{F} : \mathrm{End}(\mathcal{H}_A) \mapsto \mathrm{End}(\mathcal{H}_B)$.
A naive approach is to send the same state through each of the channels and see how similar the output states are,

$$d(\mathcal{E}, \mathcal{F}) = \max_{\rho_A} \delta(\mathcal{E}(\rho_A), \mathcal{F}(\rho_A)), \tag{1}$$

where $\delta(\rho, \sigma)$ is the trace distance between states.
However, we may want to consider that ρ_A may be entangled with some other system, because a channel that acts locally may produce global changes on the total state (e.g. break the entanglement). The stabilized distance (sometimes called the diamond norm) takes that into account:

$$d^{\diamond}(\mathcal{E}, \mathcal{F}) = \max_{\rho_{AR}} \delta(\mathcal{E} \otimes \mathcal{I}(\rho_{AR}), \mathcal{F} \otimes \mathcal{I}(\rho_{AR})), \tag{2}$$

where $\mathcal{I}$ is the identity map.



a) Show that in general $d(\mathcal{E}, \mathcal{F}) \leq d^{\diamond}(\mathcal{E}, \mathcal{F})$.

b) Compute and compare $d(\mathcal{E}, \mathcal{F})$ and $d^{\diamond}(\mathcal{E}, \mathcal{F})$, where $\mathcal{E}$ and $\mathcal{F}$ act on ρ as $\mathcal{E}_{\mathcal{A}}(\rho_A) = \mathcal{I}(\rho_A)$ and $\mathcal{F}_{\mathcal{A}}(\rho_A) = \frac{\mathbb{I}_A}{d_A}$.

Exercise 7.2 Bell-type Experiment *(6 points)*

$$\boxed{\text{A}} \quad \longleftarrow |\psi^+\rangle \longrightarrow \quad \boxed{\text{B}}$$

Consider a 2-qubit Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with basis $\{|00\rangle, |01\rangle|10\rangle, |11\rangle\}$ in the Bell-state

$$|\psi^+\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right). \tag{3}$$

Two parties, Alice and Bob, get half of state $|\psi^+\rangle$ so that Alice has qubit A and Bob has qubit B. The POVM corresponding to a measurement can be written in function of the angle α that the measurement basis makes with the $\{|0\rangle, |1\rangle\}$ basis,

$$M^{\alpha} = \left\{ |\alpha\rangle\langle\alpha|, |\alpha^{\perp}\rangle\langle\alpha^{\perp}| \right\}, \qquad |\alpha\rangle = \cos\frac{\alpha}{2}\ |0\rangle + \sin\frac{\alpha}{2}\ |1\rangle, \qquad |\alpha^{\perp}\rangle = \sin\frac{\alpha}{2}\ |0\rangle + \cos\frac{\alpha}{2}\ |1\rangle,$$

where the 1/2 factor comes from the Bloch sphere notation. We label the outcomes $+$ for $|\alpha\rangle$ and $-$ for $|\alpha^{\perp}\rangle$.

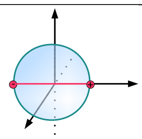
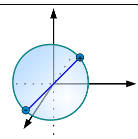
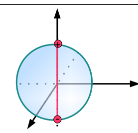
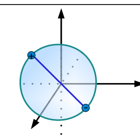
Suppose Alice performs such a measurement $\mathcal{M}^{\alpha}$ on her qubit.

a) Find the description Bob would give to his partial state on B after he knows that Alice performed the measurement M_A^{α} on A. What description would Alice give to ρ_B given that she knows what measurement outcome she received?

b) If Bob does the measurement $\mathcal{M}_B^0 = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on B, what is the probability distribution for his outcomes, P_B? How would Alice describe his probability distribution, $\mathrm{P}_{B|A}$?

c) In part a) and b) Alice and Bob have different descriptions of the quantum state ρ_B and probability distribution of measurement outcomes on that state. Explain how this subjective assignment of the scenarios at B does not contradict with the actual measurement outcomes Bob will get after doing the measurement $\mathcal{M}_B^0$.

From now on look at the case where Alice and Bob can choose two different bases each:

meas.	0	1	2	3
				
Alice	$\alpha = 0, \begin{cases} \lvert 0\rangle & \to + \\ \lvert \pi\rangle & \to - \end{cases}$		$\alpha = \frac{\pi}{2}, \begin{cases} \lvert \frac{\pi}{2}\rangle & \to + \\ \lvert \frac{-\pi}{2}\rangle & \to - \end{cases}$	
Bob		$\alpha = \frac{\pi}{4}, \begin{cases} \lvert \frac{\pi}{4}\rangle & \to + \\ \lvert \frac{5\pi}{4}\rangle & \to - \end{cases}$		$\alpha = \frac{3\pi}{4}, \begin{cases} \lvert \frac{3\pi}{4}\rangle & \to + \\ \lvert \frac{-\pi}{4}\rangle & \to - \end{cases}$

d) The joint probabilities $P_{XY|ab}(x,y)$ of Alice and Bob obtaining outcomes x and y when they measure $A = a$ and $B = b$ are given by

Alice / Bob		A =0, +	A =0, −	A =2, +	A =2, −
B=1	+	$\frac{1}{2} - \epsilon$	ϵ	$\frac{1}{2} - \epsilon$	ϵ
	−	ϵ	$\frac{1}{2} - \epsilon$	ϵ	$\frac{1}{2} - \epsilon$
B=3	+	ϵ	$\frac{1}{2} - \epsilon$	$\frac{1}{2} - \epsilon$	ϵ
	−	$\frac{1}{2} - \epsilon$	ϵ	ϵ	$\frac{1}{2} - \epsilon$

with $\epsilon = \frac{1}{2}\sin^2(\pi/8) \approx 0.07$.

Compute

$$I_N(P_{XY|AB}) = P(X = Y|A = 0, B = 3) + \sum_{|a-b|=1} P(X \neq Y|A = a, B = b).$$

This quantity, similar to a Bell inequality, captures non-locality of quantum correlations: classically, it is at least equal to 1, whereas for quantum correlations it can be smaller than 1.

e) Correlations of the above form that exist within quantum theory cannot be created classically. However, they are not the most general distributions we could consider if we are only contained by the no-signalling principle: there are in fact other joint distributions that cannot be obtained by measurements on a quantum state, but that nonetheless would not allow for instantaneous information transmission over distance (signalling). To see this, look at the following joint probability distribution, a so-called PR box:

Alice / Bob		A =0, +	A =0, −	A =2, +	A =2, −
B=1	+	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	−	0	$\frac{1}{2}$	0	$\frac{1}{2}$
B=3	+	0	$\frac{1}{2}$	$\frac{1}{2}$	0
	−	$\frac{1}{2}$	0	0	$\frac{1}{2}$

Show that the PR box

(i) is non-signalling: $P(X|a, b_1) = P(X|a, b_2), \forall a$;

(ii) is non-local: $P_{XY|ab} \neq P_{X|a}P_{Y|b}$;

(iii) yields $I_N(P_{XY|AB}) = 0$.

f) We shall now see how the above quantum correlation (coming from the Bell state) can be simulated using such a PR box combined with deterministic strategies. Imagine that Alice and Bob are given:

- with probability $1 - p$ a PR-box;
- with probability $p/4$, one of four deterministic boxes, that always outcome $++$, $+-$, $-+$ and $--$ respectively.

Find p so that the final joint probability distribution equals the one of the Bell state given above.

Extra: Suppose that Alice and Bob are allowed to perform more measurements with closer angles (so that the overlap between two consecutive bases is larger). What happens to p?