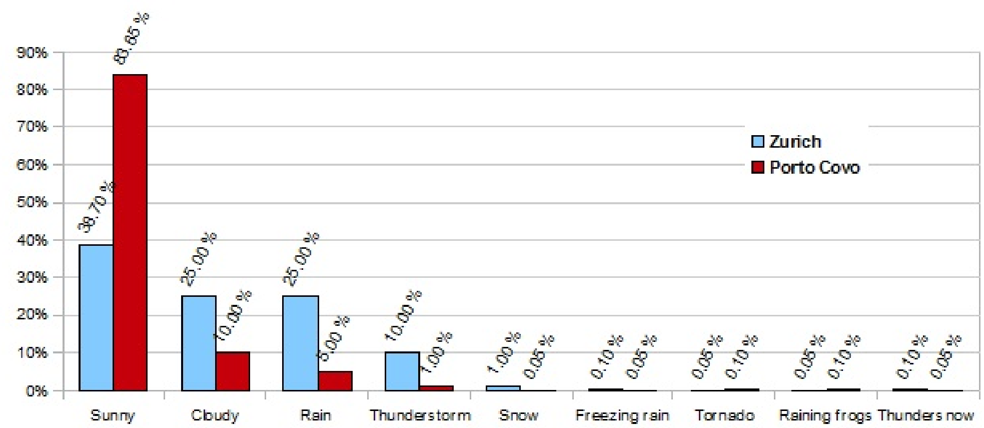
Quantum Information Theory
Series 2

Exercise 2.1 Entropy as a measure of uncertainty

These two graphs represent the probability distributions of the weather conditions for a summer day in Zurich and Porto Covo, Portugal. We will try to quantify the uncertainty we have about the weather in both cases using some entropy measures. Here $\log \equiv \log_2$.



a) Suppose you want to a make list of all the weather possibilities in both places (for instance, to decide how many different sets of clothes you need when visiting those places, to be on the safe side). How long would the two lists be?

Realistically, you do not expect snow in Porto Covo or tornados in Zurich, so you can safely leave those possibilities out of your lists if you allow for a very small error tolerance. How long are the lists if you dismiss very unlikely events (pick a reasonable definition of "unlikely")? Relate these results to the max-entropy,

$$H_{\max}(X)_P = \log |P_X|,$$

where $|P_X|$ is the size of the support of P_X (ie the number of outcomes with non-zero probability), and to its smooth version,

$$H_{\max}^{\epsilon}(X)_P = \min_{Q_X \in \mathcal{B}^{\epsilon}(P_X)} H_{\max}(X)_Q,$$

where the minimum goes over all probability distributions Q_X that are ϵ-close to P_X according to the trace distance.

b) How likely are you to correctly guess the weather in each place? Relate that to the classical min-entropy of a probability distribution P_X over $\mathcal{X}$, defined as

$$H_{\min}(X)_P = -\log \max_{x \in \mathcal{X}} P_X(x).$$

Exercise 2.2 Mutual Information

After being unsuccessful at convincing your Scottish grandfather about whether listening to the radio forecast would help to predict the weather, you have been studying information theory compulsively to try to come up with a clever argument that would make him stop mocking you. You are convinced that even though you did not guess correctly more often than he, you somehow have more *information* about the weather than he does.

a) The mutual information between two random variables is given by

$$I(X:Y)_P = H(X)_P - H(X|Y)_P,$$

where $H(X)$ is the Shannon entropy of X,

$$H(X)_P = \langle -\log P_X(x) \rangle_x = -\sum_x P_X(x) \log P_X(x)$$

and $H(X|Y)$ is the conditional Shannon entropy of X given Y,

$$H(X|Y)_P = \langle -\log P_{X|Y=y}(x) \rangle_{x,y} = -\sum_{x,y} P_{XY}(x,y) \ \log P_{X|Y=y}(x) = H(XY)_P - H(Y)_P.$$

Compute the mutual information between your guess and the actual weather, and do the same for your grandfather. Remember that your grandfather knows it rains on 80% of the days. You also listen to the forecast, knowing it is right 80% of the time and always correct when it predicts rain. (See the solutions to Exercise Sheet 1 for the other probabilities).

b) You read about a betting game that you think might help you convince your grandfather about the weather forecast (see Cover and Thomas, Chapter 6). It is defined as follows:

Assume that M horses are in a race. The ith horse will win with probability p_i. You can bet some fraction of your money on each horse b_i, and you must bet all of your money (so $\sum_i b_i = 1$). If horse i wins, you will make $b_i g_i$ amount of money, and you get nothing if horse i loses. Therefore, g_i is the proportion of your bet money that you will win. The factor by which you wealth increases (that is, the total money you have after N rounds of betting on races relative to the initial wealth) is:

$$S_N = \prod_{i=1}^{N} S(X_i),$$

where $S(X) = b_X g_X$ is the factor by which your wealth is multiplied when horse X wins.

i) Assuming that the race outcomes, your betting strategy and the payout are i.i.d., and considering the asymptotics for $N \to \infty$, show that the *doubling rate* is

$$W(b,p) = E(\log S(X)) = \sum_{k=1}^{M} p_k \log b_k g_k,$$

where $S_N = 2^{nW(b,p)}$.

ii) Show that the optimal asymptotic doubling rate is

$$W_{opt}(p) = \sum_i p_i \log(g_i) - H(p).$$

iii) How does the strategy in (bii) differ from the strategy to optimize your expected wealth after a single bet?

c) You would now like to use this newly found knowledge of betting games to show your grandfather that you have more information about the weather. You and your grandfather start with £1. Every night each of you can bet some proportion b_R and b_S of your money on whether it will be rainy of sunny the next day. You must both bet all of your money each night. If your guess was right you double the amount you bet. All winnings will be used in future rounds.

Assuming you would repeat this many, many times, in independent rounds, what would your strategy be? And your grandfather's?

d) Applying the strategy from c), what is the expected wealth for each of you after the first round?

e) Still applying the strategy from c), what is the probability that your grandfather has more money than you after the first round?