Quantum Information Theory
Series 1

Exercise 1.1 Trace distance

The trace distance (or L_1-distance) between two probability distributions P_X and Q_X over a discrete alphabet $\mathcal{X}$ is defined as

$$\delta(P_X, Q_X) = \frac{1}{2} \sum_{x \in \mathcal{X}} |P_X(x) - Q_X(x)| \,. \tag{1}$$

The trace distance may also be written as

$$\delta(P_X, Q_X) = \max_{\mathcal{S} \subseteq \mathcal{X}} |P_X[\mathcal{S}] - Q_X[\mathcal{S}]| \,, \tag{2}$$

where we maximise over all events $S \subseteq \mathcal{X}$ and the probability of an event is given by $P_X[\mathcal{S}] = \sum_{x \in \mathcal{S}} P_X(x)$.

a) Show that $\delta(\cdot, \cdot)$ is a good measure of distance by proving that $0 \leq \delta(P_X, Q_X) \leq 1$ and the triangle inequality $\delta(P_X, R_X) \leq \delta(P_X, Q_X) + \delta(Q_X, R_X)$ for arbitrary probability distributions P_X, Q_X and R_X.

b) Show that definitions (2) and (1) are equivalent.

c) Let us now find an operational meaning for the trace distance. Suppose that P_X and Q_X represent the probability distributions of the outcomes of two dice that look identical. You are only allowed to throw one of them once and then you have to guess which die it was. What is your best strategy? What is the probability that you guess correctly and how can you relate this probability to the trace distance $\delta(P_X, Q_X)$?

Exercise 1.2 Weak law of large numbers

Let A be a positive random variable with expectation value $\langle A \rangle = \sum_a a \ P_A(a)$. Let $P[A \geq \varepsilon]$ denote the probability of an event $\{A \geq \varepsilon\}$.

a) Prove Markov's inequality

$$P[A \geq \varepsilon] \leq \frac{\langle A \rangle}{\varepsilon}. \tag{3}$$

b) Use Markov's inequality to prove the weak law of large numbers for i.i.d. X_i:

$$\lim_{n \to \infty} P \left[\left(\frac{1}{n} \sum_i X_i - \mu \right)^2 \geq \varepsilon \right] = 0 \quad \text{for any } \varepsilon > 0, \mu = \langle X_i \rangle. \tag{4}$$

Exercise 1.3 Conditional probabilities: how knowing more does not always help

Suppose you are visiting your grandfather in Scotland. You gave him a radio for Christmas, but he does not like such modern technology and has not used it. You decide to initiate a game to prove to him that this technology is useful: every evening you listen to the weather forecast alone on the radio and then both you and your grandfather try to guess if it will rain next day. Having lived there since birth, your grandfather knows that it rains on 80% of the days. You have reached the same conclusion on previous summer holidays. You also know that the weather forecast is right 80% of the time and is always correct when it predicts rain.

a) What is the optimal strategy for your grandfather? What is your optimal strategy?

b) Both of you keep a record of your guesses and the actual weather for statistical analysis. After some months, who will have guessed correctly more often?

c) Can you design a method to convince your grandfather that the forecast is useful? Be precise.