# Quantum Field Theory I — Problem Set 1

ETH Zurich, HS12 — G. Abelof, J. Cancino, F. Dulat, B. Mistlberger, Prof. N. Beisert

## 1. Classical particle in an electromagnetic field

Consider the classical Lagrangian density of a particle of mass $m$ and charge $q$, moving in an electromagnetic field, specified by the electric potential $\phi(x)$ and the magnetic vector potential $A_i(x)$:

$$\mathcal{L} = \frac{m}{2}(\partial_t x^i)^2 + qA_i(x)\partial_t x^i - q\phi(x). \tag{1}$$

Find

**a)** The canonical momentum conjugate to the coordinate $x_i$.

**b)** The equations of motion corresponding to the Lagrangian density.

**c)** The Hamiltonian of the system

Compare your results to a free particle.

## 2. Stress-energy tensor

Consider the variational principle:

$$\delta S = 0 = \delta \int d^4x \, \mathcal{L}(\phi, \pi). \tag{2}$$

The Lagrangian density $\mathcal{L}$ is a function of the two classical fields $\phi(x)$ and $\pi_\mu(x) = \partial_\mu \phi(x)$. Note that $\mathcal{L}$ does not depend directly on the space-time coordinate $x^\mu$, but only indirectly through $\phi$ and $\pi$. Show that the conserved Noether current associated with infinitesimal space-time translations

$$x^\mu \to x^\mu + \epsilon^\mu \tag{3}$$

is the stress-energy tensor $T^{\mu\nu}$ given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \pi_\mu}\partial^\nu \phi - g^{\mu\nu}\mathcal{L}. \tag{4}$$

Remind yourself how a general function $f(x^\mu)$ of the space-time coordinates will transform under an infinitesimal translation.
Note that $x^\mu$ is a standard Minkowski-space coordinate, so that $x^0$ is the time.
$\eta^{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ is the metric tensor.

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## 3. Coherent quantum oscillator

Consider the Hamiltonian of a quantum harmonic oscillator:

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} . \tag{5}$$

**a)** Introduce ladder operators to diagonalise the Hamiltonian.

**b)** Calculate the expectation values of the number operator $N \sim a^\dagger a$ as well as of the $x$ and $p$ operator in a general number state $|n\rangle$.

**c)** Calculate the variances $\Delta x$, $\Delta p$ and $\Delta N$ in the same state $|n\rangle$ and use them to determine the Heisenberg uncertainty of $|n\rangle$.

**d)** Show that the coherent state

$$|\alpha\rangle = e^{\alpha p}|0\rangle \tag{6}$$

is an eigenstate of the annihilation operator you defined in a).

**e)** Calculate the time-dependent expectation values of $x$, $p$ and $N$:

$$\langle\alpha|x(t)|\alpha\rangle \tag{7}$$
$$\langle\alpha|p(t)|\alpha\rangle \tag{8}$$
$$\langle\alpha|N(t)|\alpha\rangle \tag{9}$$

as well as the corresponding variances to determine the uncertainty of the state $|\alpha\rangle$. Compare your result with the result obtained in c).

## 4. Relativistic point particle

The action of a relativistic point particle is given by

$$S = -\alpha \int_{\mathcal{P}} ds \tag{10}$$

with the relativistic line element

$$ds^2 = -g_{\mu\nu} dX^\mu dX^\nu = dt^2 - dx^2 - dy^2 - dz^2 \tag{11}$$

and $\alpha$ a (yet to be determined) constant.
The path $\mathcal{P}$ between two points $X_1^\mu$ and $X_2^\mu$ can be parametrised by a parameter $\tau$. With that, the integral of the line element $ds$ becomes an integral over the parameter

$$S = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-g_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}} . \tag{12}$$

**a)** Parametrise the path by the time coordinate $t$ (i.e. $x^0$) and take the non-relativistic limit $|\partial_0 x^\mu| \ll 1$ to determine the value of the constant $\alpha$.

**b)** Derive the equations of motion by varying the action. *Hint:* You may want to determine the canonically conjugate momentum first.

# Quantum Field Theory I — Problem Set 2

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## 1. Causality

Consider a scalar field $\phi(x)$ as defined in the lecture. First we want to calculate the amplitude

$$\Delta_+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle \tag{1}$$

for a particle to propagate from point $x$ to point $y$.

**a)** Calculate $\Delta_+(x-y)$ for time-like separation, i.e. $x^0 - y^0 = t$, $x^i - y^i = 0$.

**b)** Calculate $\Delta_+(x-y)$ for space-like separation, i.e. $x^0 - y^0 = 0$, $x^i - y^i = r^i$.

The next thing we need to check is whether a measurement at $x$ can affect another measurement at $y$. To do this one computes the commutator $[\phi(x), \phi(y)]$. If it vanishes, the two measurements cannot affect each other and causality is preserved.

**c)** Show that the commutator vanishes for a space-like separation of $x$ and $y$.

## 2. Complex scalar field

We want to investigate the theory of a complex scalar field $\phi = \phi(x)$. The theory is described by the Lagrangian (density):

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{2}$$

As a complex scalar field has two degrees of freedom, we can treat $\phi$ and $\phi^*$ as independent fields with one degree of freedom each.

**a)** Find the conjugate momenta $\pi(\vec{x})$ and $\pi^*(\vec{x})$ to $\phi(\vec{x})$ and $\phi^*(\vec{x})$ and the canonical commutation relations. (Note: we choose $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ rather than $\pi = \partial\mathcal{L}/\partial\dot{\phi}^*$.)

**b)** Find the Hamiltonian of the theory.

**c)** Introduce creation and annihilation operators to diagonalise the Hamiltonian.

**d)** Show that the theory contains two sets of particles of mass $m$.

**e)** Consider the conserved charge

$$Q = -\frac{i}{2}\int d^3\vec{x}\,(\pi\phi - \phi^*\pi^*). \tag{3}$$

Rewrite it in terms of ladder operators and determine the charges of the two particle species.

## 3. Momentum

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{4}$$

of a real scalar field $\phi = \phi(x)$.

**a)** Write down the stress-energy tensor of the theory using the general result obtained in the previous exercise sheet.

**b)** Derive

$$P^\mu = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})}\, p^\mu(\vec{p})\, a^\dagger(\vec{p}) a(\vec{p}) \tag{5}$$

starting from $P^\mu = \int d^3\vec{x}\, T^{0\mu}$.

**c)** Calculate the commutator $[P^\mu, \phi(x)]$ and interpret the result.

# Quantum Field Theory I — Problem Set 3

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## 1. The Feynman propagator for a real scalar field

Consider a real scalar field $\phi(x)$.

**a)** Use the Fourier expansion of $\phi(x)$ to show that

$$\mathit{\Delta}_+(x) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})} \exp\bigl(-ie(\vec{p})t - i\vec{p}\cdot\vec{x}\bigr) \tag{1}$$

with $e(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}$.

**b)** Use Cauchy's residue theorem to show that $\mathit{\Delta}_+(x)$ can be also written as

$$\mathit{\Delta}_+(x) = i \int_{C_+} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 + m^2}\,, \tag{2}$$

where the integration over the contour $C_+$, which is given in the left figure below, corresponds to the (complex) variable $p_0$.

The Feynman propagator for the real scalar field is defined as

$$G_{\mathrm{F}}(x-y) = i\theta(x^0 - y^0)\mathit{\Delta}_+(x-y) + i\theta(y^0 - x^0)\mathit{\Delta}_+(y-x). \tag{3}$$

**c)** Show that it satisfies the defining relation for a propagator

$$(-\partial^2 + m^2)G_{\mathrm{F}}(x) = \delta^{d+1}(x). \tag{4}$$

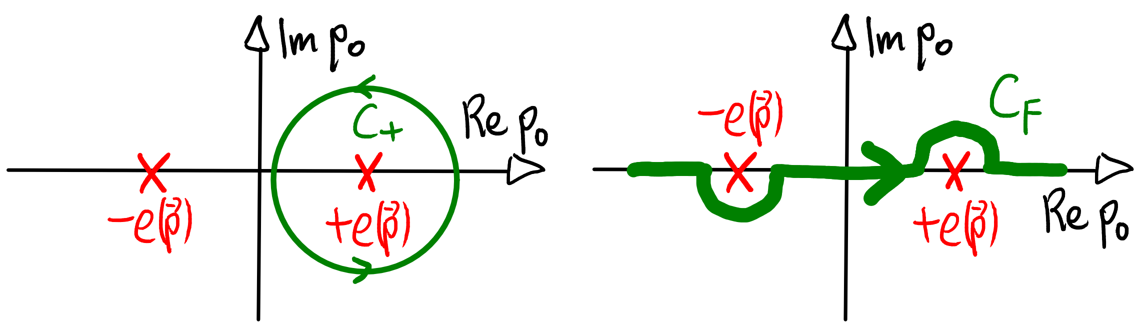
**d)** Show that

$$G_{\mathrm{F}}(x) = \int_{C_{\mathrm{F}}} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 + m^2}\,, \tag{5}$$

with the contour $C_{\mathrm{F}}$ given in the right figure below.

**e)** Show that the integral in eq.(5) is equivalent to an integral over the real axis

$$G_{\mathrm{F}}(x) = \int_{\mathbb{R}} \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot x}}{p^2 + m^2 - i\varepsilon}\,. \tag{6}$$

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## 2. Conservation of charge with complex scalar fields

Consider a free complex scalar field described by

$$\mathcal{L} = -(\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \tag{7}$$

**a)** Show that the transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{i\alpha}\phi(x) \tag{8}$$

leaves the Lagrangian density invariant.

**b)** Find the conserved current associated with this symmetry.

If we now consider two complex scalar fields, the Lagrangian density is given by

$$\mathcal{L} = -(\partial_\mu \phi_a^*)(\partial^\mu \phi^a) - m\phi_a^* \phi^a \qquad a = 1, 2. \tag{9}$$

**c)** Show that

$$\phi^a(x) \longrightarrow \phi'^a(x) = M^a{}_b \phi^b(x) \tag{10}$$

with $M \in U(2) = \left\{A \in \mathbb{C}^{2\times 2} : A^{-1} = A^\dagger = (A^*)^{\mathrm{T}}\right\}$ is a symmetry transformation.

**d)** Show that now there are four conserved charges, one given by the generalisation of part b), and the other three given by

$$Q_i = \frac{i}{2}\int d^3\vec{x}\,\left(\phi_a^*(\sigma^i)^a{}_b \pi^{*\,b} - \pi_a(\sigma^i)^a{}_b \phi^b\right), \tag{11}$$

where $\sigma^i$ are the Pauli matrices.

## 3. Symmetry of the stress-energy tensor

Consider a relativistic scalar field theory specified by some Lagrangian $\mathcal{L}(\phi, \partial\phi)$.

**a)** Compute the variation of $\mathcal{L}(\phi(x), \partial\phi(x))$ under infinitesimal Lorentz transformations (note: $\omega^{\mu\nu} = -\omega^{\nu\mu}$)

$$x^\mu \longrightarrow x^\mu - \omega^\mu{}_\nu x^\nu. \tag{12}$$

**b)** Assuming that $\mathcal{L}(x)$ transforms as a scalar field, i.e. just like $\phi(x)$, derive another expression for its variation under Lorentz transformations.

**c)** Compare the two expressions to show that the two indices of the stress-energy tensor are symmetric

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = T^{\nu\mu}. \tag{13}$$

# Quantum Field Theory I — Problem Set 4

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## 1. Properties of $\gamma$-matrices

The $\gamma$-matrices satisfy a *Clifford algebra*,[1]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbf{1}. \tag{1}$$

**a)** Show the following contraction identities using (1):

1. $\gamma^\mu\gamma_\mu = -4 \cdot \mathbf{1}$.
2. $\gamma^\mu\gamma^\nu\gamma_\mu = 2\gamma^\nu$.
3. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}\mathbf{1}$.
4. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = 2\gamma^\sigma\gamma^\rho\gamma^\nu$.

**b)** Show the following trace properties using (1):

1. $\operatorname{tr}\gamma^{\mu_1}\cdots\gamma^{\mu_n} = 0$ if $n$ is odd.
2. $\operatorname{tr}\gamma^\mu\gamma^\nu = -4\eta^{\mu\nu}$.
3. $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$.

## 2. Dirac and Weyl representations of the $\gamma$-matrices

Using the Pauli matrices together with the identity,

$$\sigma^0 \equiv \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2}$$

we can realize the *Dirac representation* of the $\gamma$-matrices,

$$\gamma_{\mathrm{D}}^0 \equiv \sigma^0 \otimes \sigma^3, \qquad \gamma_{\mathrm{D}}^j \equiv \sigma^j \otimes i\sigma^2 \quad (j = 1, 2, 3), \tag{3}$$

where

$$A \otimes B \equiv \begin{pmatrix} b_{11}A & b_{12}A\\ b_{21}A & b_{22}A\end{pmatrix}. \tag{4}$$

Denoting the Pauli matrices collectively by $\sigma^\mu$ and defining $(\bar\sigma^0, \bar\sigma^i) = (\sigma^0, -\sigma^i)$. we can then define the $\gamma$-matrices in the *Weyl representation*:

$$\gamma_{\mathrm{W}}^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}. \tag{5}$$

Show that both representations satisfy the Clifford algebra (1). Can you show their equivalence, i.e. $\gamma_{\mathrm{W}}^\mu = T\gamma_{\mathrm{D}}^\mu T^{-1}$ for some matrix $T$?

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[1] The minus sign is due to our choice of metric $\eta^{\mu\nu} = \operatorname{diag}(-1, +1, +1, +1)$! Alternatively, we might use a plus sign (as in the opposite signature) and instead multiply all $\gamma$-matrices by a factor of $i$.

## 3. Spinors, spin sums and completeness relations

In this exercise we will use the Weyl representation (5) defined in the previous exercise.

**a)** Show that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = -p^2$.

**b)** Prove that the below 4-spinor $u_s(\vec{p})$ solves Dirac's equation $(p_\mu \gamma^\mu - m\mathbf{1})u_s(\vec{p}) = 0$

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi_s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi_s \end{pmatrix}, \tag{6}$$

where $\xi_\pm$ form a basis of 2-spinors.

**c)** Suppose, the 2-spinors $\xi_+$ and $\xi_-$ are orthonormal. What does it imply for $\xi_s^\dagger \xi_s$ and

$$\sum_{s \in \{+,-\}} \xi_s \xi_s^\dagger \,? \tag{7}$$

**d)** Show that $\bar{u}_s(\vec{p}) u_s(\vec{p}) = 2m$ for $s \in \{+,-\}$.

**e)** Show the *completeness relation*:

$$\sum_{s \in \{+,-\}} u_s(\vec{p}) \bar{u}_s(\vec{p}) = p_\mu \gamma^\mu + m\mathbf{1}. \tag{8}$$

## 4. Gordon identity

Prove the *Gordon identity*,

$$\bar{u}_t(\vec{q}) \gamma^\mu u_s(\vec{p}) = \frac{1}{2m}\, \bar{u}_t(\vec{q}) \big[ -(q+p)^\mu - \tfrac{1}{2}[\gamma^\mu, \gamma^\nu](q-p)_\nu \big] u_s(\vec{p}). \tag{9}$$

*Hint:* You can do this using just (1).

## 1. Spinor rotations

The Dirac equation is invariant under Lorentz transformations $\Psi'(x') = S\Psi(x)$ if the spinor transformation matrix $S$ satisfies

$$\Lambda^\mu{}_\nu \, S^{-1}\gamma^\nu S = \gamma^\mu. \tag{1}$$

For an infinitesimal Lorentz transformation $\Lambda_{\mu\nu} = \eta_{\mu\nu} + \delta\omega_{\mu\nu}$ this is fulfilled if

$$S = 1 + \tfrac{1}{8}\delta\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]. \tag{2}$$

**a)** Find the infinitesimal spinor transformation $\delta S$ for a rotation around the 3-axis, i.e. only $\delta\omega_{12} = -\delta\omega_{21} \neq 0$.

**b)** Finite transformations are obtained by considering a consecutive application of infinitely many, $N \to \infty$, infinitesimal transformations $\delta\omega = \omega/N$

$$S = \lim_{N\to\infty}\left(1 + \frac{1}{8}\frac{\omega_{\mu\nu}}{N}[\gamma^\mu, \gamma^\nu]\right)^N = \exp\left(\tfrac{1}{8}\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right). \tag{3}$$

Compute the finite rotation with angle $\omega_{12}$ around the same axis as before. Also compute the finite transformation $\Lambda = \exp(\omega)$ for vectors.

**c)** What happens to the individual components of a spinor under this transformation? What is the period of the transformation in the angle $\omega_{12}$? Compare it to the finite rotation for vectors.

## 2. Completeness for gamma matrices

An arbitrary product of $\gamma$-matrices is proportional to one of the following 16 linearly independent matrices $\gamma^a$ (here $a$ is a multi-index which specifies the type of matrix, $\mathrm{S}, \mathrm{P}, \mathrm{V}, \mathrm{A}, \mathrm{T}$, along the corresponding indices if any)

- $\Gamma^{\mathrm{S}} = 1$,
- $\Gamma^{\mathrm{P}} = \gamma^5$,
- $\Gamma^{\mathrm{V},\mu} = \gamma^\mu$,
- $\Gamma^{\mathrm{A},\mu} = i\gamma^5\gamma^\mu$,
- $\Gamma^{\mathrm{T},\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

**a)** Show that the trace of any product of $\Gamma$ matrices is given by $\mathrm{tr}(\Gamma^a\Gamma^b) = 4\delta^{ab}$. For simplicity we ignore the signs arising from the Lorentz signature.

**b)** Show that for any $a \neq b$ there is a $n \neq \mathrm{S}$ such that $\Gamma^a\Gamma^b = \alpha\Gamma^n$ with some $\alpha \in \mathbb{C}$.

**c)** Show that the matrices are linearly independent and therefor form a complete basis of $4\times 4$ spinor matrices. *Hint:* To do this consider a sum $\sum_a \alpha_a\Gamma^a = 0$. What can be said about the coefficients?

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## 3. Fierz identity

**a)** Use the linear independence of the $\Gamma^a$ matrices to show that

$$\delta^\alpha_\gamma \delta^\beta_\delta = \sum_i \frac{1}{4} (\Gamma_i)^\alpha{}_\delta (\Gamma_i)^\beta{}_\gamma . \tag{4}$$

*Hint:* Decompose an arbitrary matrix $M^\alpha{}_\beta = \sum_i m_i (\Gamma^i)^\alpha{}_\beta$ and find the coefficients $m_i$.

**b)** Use the result from a) to show the Fierz identity:

$$(\Gamma^i)^\alpha{}_\beta (\Gamma^j)^\gamma{}_\delta = \sum_{k,l} \frac{1}{16} \operatorname{tr}(\Gamma^i \Gamma^l \Gamma^j \Gamma^k) (\Gamma^k)^\alpha{}_\delta (\Gamma^l)^\gamma{}_\beta . \tag{5}$$

**c)** Find the Fierz transformation for the spinor products

- $(\bar{u}_1 u_2)(\bar{u}_3 u_4)$,
- $(\bar{u}_1 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_4)$.

# Quantum Field Theory I — Problem Set 6

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## 1. Helicity and Chirality

In four dimensions we can define the chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{1}$$

**a)** Show that $\gamma^5$ satisfies

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \tag{2}$$

$$\left(\gamma^5\right)^2 = 0. \tag{3}$$

**b)** The helicity operator $h(\vec{p})$ is defined as

$$h(\vec{p}) = \frac{1}{|\vec{p}|}\begin{pmatrix} \sigma^i p_i & 0 \\ 0 & \sigma^i p_i \end{pmatrix}. \tag{4}$$

Show that helicity and chirality are equivalent for a massless spinor $u_s(\vec{p})$.

**c)** Consider the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi. \tag{5}$$

Find the corresponding Hamiltonian.

**d)** Show that chirality is not conserved for a massive fermion. *Hint:* You do not need to compute the time-evolution, just show that it is non-trivial.

**e)** Show that helicity is conserved but not Lorentz invariant.

**f)** Show that the Dirac Lagrangian is invariant under a chiral transformation $U = \exp(-i\alpha\gamma^5)$ of the fields for $m = 0$, and derive the associated conserved current. Show that having a non-zero mass breaks the symmetry.

## 2. Discrete symmetries

Recall the $\Gamma^X$ matrices from the last exercise sheet. Using these products of $\gamma$ matrices, we can define the following bilinears:

$$S = \bar{\psi}\Gamma^S\psi, \tag{6}$$

$$P = \bar{\psi}\Gamma^P\psi, \tag{7}$$

$$V^\mu = \bar{\psi}\Gamma^{V,\mu}\psi, \tag{8}$$

$$A^\mu = \bar{\psi}\Gamma^{A,\mu}\psi, \tag{9}$$

$$T^{\mu\nu} = \bar{\psi}\Gamma^{T,\mu\nu}\psi. \tag{10}$$

**a)** Calculate their behaviour under parity transformations $P(\psi(t,\vec{x})) = \gamma^0\psi(t,-\vec{x})$.

**b)** Show that the Dirac Lagrangian (5) is invariant under $CPT$ as well as under $P$.

**c)** Starting from the Dirac Lagrangian, write down a similar Lagrangian that is $CPT$ invariant but not $P$ invariant.

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## 3. Electrodynamics

Consider the Lagrange density

$$\mathcal{L}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\,, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{11}$$

and $J^\mu$ is some external source field.

**a)** Show that the Euler–Lagrange equations are the inhomogenous Maxwell equations. The usual electromagnetic fields are defined by $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. Are these all Maxwell equations?

**b)** Construct the stress-energy tensor for this theory.

**c)** Convince yourself that the stress-energy tensor is not symmetric. In order to make it symmetric consider

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu,\nu}\,, \tag{12}$$

where $K^{\lambda\mu\nu}$ is anti-symmetric in the first two indices. By taking

$$K^{\lambda\mu,\nu} = F^{\mu\lambda}A^\nu \tag{13}$$

show that the modified stress energy tensor $\hat{T}^{\mu\nu}$ is symmetric, and that it leads to the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \tfrac{1}{2}(\vec{E}^{\,2} + \vec{B}^{\,2})\,, \qquad \vec{\mathcal{S}} = \vec{E}\times\vec{B}\,. \tag{14}$$

**d)** *For fun:* Show that all of Maxwell's equations can summarised as

$$\gamma^\nu\gamma^\rho\gamma^\sigma\partial_\nu F_{\rho\sigma} = -2\gamma^\nu J_\nu. \tag{15}$$

## 1. Polarisation vectors of a massless vector field

Each Fourier mode in the plane wave expansion of a massless vector field has the form

$$A_\mu^{(\lambda)}(\vec{p};x) = N(\vec{p})\,\epsilon_\mu^{(\lambda)}(\vec{p})\,e^{ip\cdot x} \tag{1}$$

Without any loss of generality the polarisation vectors $\epsilon_\mu^{(\lambda)}(\vec{p})$ can be chosen to form a four-dimensional orthonormal system satisfying

$$\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon^{(\kappa)\mu}(\vec{p}) = \eta^{\lambda\kappa}. \tag{2}$$

**a)** Show that the following choice satisfies (2)

$$\epsilon_\mu^{(0)}(\vec{p}) = n_\mu, \tag{3}$$
$$\epsilon_\mu^{(1)}(\vec{p}) = (0, \vec{\epsilon}^{\,(1)}(\vec{p})), \tag{4}$$
$$\epsilon_\mu^{(2)}(\vec{p}) = (0, \vec{\epsilon}^{\,(2)}(\vec{p})), \tag{5}$$
$$\epsilon_\mu^{(3)}(\vec{p}) = \left(p_\mu + n_\mu(p\cdot n)\right)/|p\cdot n|, \tag{6}$$

where $n_\mu = (1,0)$ and $\vec{p}\cdot\vec{\epsilon}^{\,(k)}(\vec{p}) = 0$ as well as $\vec{\epsilon}^{\,(k)}(\vec{p})\cdot\vec{\epsilon}^{\,(l)}(\vec{p}) = \delta^{kl}$.

**b)** Use the polarisation vectors to verify the completeness relation

$$\sum_{\lambda=0}^{3}\eta_{\lambda\lambda}\,\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon_\nu^{(\lambda)}(\vec{p}) = \eta_{\mu\nu}. \tag{7}$$

**c)** Show for the physical modes of the photon that

$$\sum_{\lambda=1}^{2}\epsilon_\mu^{(\lambda)}(\vec{p})\,\epsilon_\nu^{(\lambda)}(\vec{p}) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} - \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}\,. \tag{8}$$

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## 2. Spinor helicity framework

The spinor helicity framework is a method to conveniently work with massless particles and their helicity modes.
Write a momentum 4-vector $p_\mu$ as a $2 \times 2$ matrix $P$

$$P = \sigma^\mu p_\mu. \tag{9}$$

**a)** Show that the inverse transformation is given by $p_\mu = -\frac{1}{2}\,\mathrm{tr}(\bar{\sigma}_\mu P)$.

**b)** Show that $\det P = -p^2$.

**c)** Explain why the momentum $P$ of a massless particle can be expressed as a product of a (bosonic) 2-spinor $\lambda$ and its hermitian conjugate $\lambda^\dagger$

$$P = \lambda\lambda^\dagger. \tag{10}$$

Is $\lambda$ uniquely determined through $p$? What can you say about the energy $p_0$?

**d)** Show that the Lorentz-invariant integral over the light cone can be expressed as a plain integral over all $\lambda$'s

$$\int \frac{dp_1\, dp_2\, dp_3}{(2\pi)^3\, 2e(\vec{p})} f(\vec{p}) = \int \frac{d\lambda_1 d\lambda_1^*\, d\lambda_2 d\lambda_2^*}{4(2\pi)^4} f(\vec{p}(\lambda, \lambda^\dagger)). \tag{11}$$

*Hint:* As a fourth variable for the integral on the l.h.s. you may use the undetermined complex phase $\varphi = -\frac{i}{2}\log(\lambda_1/\lambda_1^*)$ of $\lambda_1$ integrated over $0 \le \varphi < 2\pi$.

Given some non-trivial 2-spinor $\mu$ (not proportional to $\lambda$), two polarisation vectors with helicity $h = \pm 1$ can be constructed as

$$\epsilon_\mu^{(+)}(\vec{p}) = \frac{\mu^\dagger \bar{\sigma}_\mu \lambda}{\mu^\dagger \sigma^2 \lambda^*}, \qquad \epsilon_\mu^{(-)}(\vec{p}) = \frac{\lambda^\dagger \bar{\sigma}_\mu \mu}{\lambda^\mathsf{T} \sigma^2 \mu}. \tag{12}$$

**e)** Show that $p \cdot \varepsilon^{(\pm)}(\vec{p}) = 0$.

**f)** Show that a change in $\mu$ acts as a gauge transformation on the polarisation vectors, i.e. $\delta\varepsilon_\mu^{(\pm)} \sim p_\mu$. *Hint:* Parametrise $\delta\mu$ as a linear combination of $\mu$ and $\lambda$.

## 3. The photon propagator with a gauge fixing term

Consider the Lagrangian for a free massless vector field modified by a gauge fixing term

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2}\xi(\partial_\lambda A^\lambda)^2. \tag{13}$$

**a)** Show that the Lagrangian is equivalent to the following one up to a total derivative

$$\mathcal{L}' = -\tfrac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - \tfrac{1}{2}(\xi - 1)(\partial_\lambda A^\lambda)^2. \tag{14}$$

**b)** Show that the equal-time commutation relations are (you may use $\mathcal{L}$ or $\mathcal{L}'$)

$$\big[A^\mu(t,\vec{x}), A^\nu(t,\vec{y})\big] = 0, \tag{15}$$

$$\big[A^\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{y})\big] = i\left(\eta^{\mu\nu} + \frac{\xi - 1}{\xi}\,\delta_0^\mu \delta_0^\nu\right)\delta^3(\vec{x} - \vec{y}), \tag{16}$$

$$\big[\dot{A}^\mu(t,\vec{x}), \dot{A}^\nu(t,\vec{y})\big] = -i\,\frac{\xi - 1}{\xi}\left(\delta_0^\mu \delta^{\nu k} + \delta_0^\nu \delta^{\mu k}\right)\partial_k \delta^3(\vec{x} - \vec{y}). \tag{17}$$

**c)** Show that the propagator is given by

$$G^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\, \frac{1}{p^2}\left(\eta^{\mu\nu} + \frac{\xi - 1}{\xi}\frac{p^\mu p^\nu}{p^2}\right). \tag{18}$$

## 1. The massive vector field

On this sheet we will develop the QFT for the free massive spin-1 field $V_\mu$.
We start with the Lagrangian for the electromagnetic field and simply add a mass term

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu V^\nu \partial_\mu V_\nu - \tfrac{1}{2}\partial^\mu V^\nu \partial_\nu V_\mu - \tfrac{1}{2}m^2 V^\mu V_\mu. \tag{1}$$

**a)** Derive the Euler–Lagrange equations of motion for $V_\mu$.

**b)** By taking a derivative of the equation, show that $V_\mu$ is a conserved current.

**c)** Show that $V_\mu$ satisfies the Klein–Gordon equation.

## 2. Hamiltonian formulation

The Hamiltonian formulation of the massive vector is somewhat tedious due to the presence of constraints.

**a)** Derive the momenta $\Pi_\mu$ conjugate to the fields $V_\mu$. Considering the space and time components separately, what do you notice?

Your observation is related to constraints. The time component $V_0$ of the vector field is completely determined by the spatial components and their conjugate momenta (without making reference to time derivatives)

**b)** Use the equations derived in problem 1 to show that

$$V_0 = -m^{-2}\partial_k \Pi_k, \qquad \dot{V}_0 = \partial_k V_k. \tag{2}$$

**c)** Substitute this solution for $V_0$ and $\dot{V}_0$ into the Lagrangian and perform a Legendre transformation to obtain the Hamiltonian. Show that

$$H = \int d^3\vec{x} \Big( \tfrac{1}{2}\Pi_k \Pi_k + \tfrac{1}{2}m^{-2}\partial_k \Pi_k \partial_l \Pi_l + \tfrac{1}{2}\partial_k V_l \partial_k V_l - \tfrac{1}{2}\partial_l V_k \partial_k V_l + \tfrac{1}{2}m^2 V_k V_k \Big). \tag{3}$$

**d)** Derive the Hamiltonian equations of motion for $V_k$ and $\Pi_k$, and compare them to the results of problem 1.

## 3. Commutators

The unequal time commutators $[V_\mu(x), V_\nu(y)] = \Delta^{\mathrm{V}}_{\mu\nu}(x-y)$ for the massive vector field read

$$\Delta^{\mathrm{V}}_{\mu\nu}(x) = \left(\eta_{\mu\nu} - m^{-2}\partial_\mu \partial_\nu\right) \Delta(x), \tag{4}$$

where $\Delta(x)$ is the corresponding function for the scalar field.

**a)** Show that these obey the equations derived in problem 1.

**b)** Show explicitly that they obey the constraint equations in 2b), i.e.

$$[m^2 V_0(x) + \partial_k \Pi_k(x), V_\nu(y)] = [\dot{V}_0(x) - \partial_k V_k(x), V_\nu(y)] = 0. \tag{5}$$

**c)** Confirm that the equal time commutators take the canonical form

$$[V_k(\vec{x}), V_l(\vec{y})] = [\Pi_k(\vec{x}), \Pi_l(\vec{y})] = 0, \qquad [V_k(\vec{x}), \Pi_l(\vec{y})] = i\delta_{kl}\delta^3(\vec{x}-\vec{y}). \tag{6}$$

$\longrightarrow$

## 4. Coupled Maxwell and Dirac fields

Quantum electrodynamics describes a coupled system of a Maxwell and an electrically charged Dirac field. The Lagrangian density for this theory is given by

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu . \tag{7}$$

**a)** Derive the classical equations of motion.

**b)** Obtain the stress-energy tensor and show that it is invariant under the gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x), \tag{8}$$

$$\psi'(x) = \exp\left[-ie\alpha(x)\right]\psi(x). \tag{9}$$

**c)** Show that the total energy and momentum of the system are conserved.

# Quantum Field Theory I — Problem Set 9

ETH Zurich, HS12 — G. Abelof, J. Cancino, F. Dulat, B. Mistlberger, Prof. N. Beisert

## 1. 4-point interaction in scalar QED

Consider a $U(1)$ gauge theory with two complex massive scalar fields $\phi, \chi$ and one vector field $A_\mu$. Each of the scalar fields is coupled to the gauge field and they both have the same coupling $e$. The Lagrangian density of the theory is given by

$$\mathcal{L} = -\tfrac{1}{2}(D_\mu\phi)^\dagger D^\mu\phi - \tfrac{1}{2}(D_\mu\chi)^\dagger D^\mu\chi - \tfrac{1}{2}m^2\phi^\dagger\phi - \tfrac{1}{2}m^2\chi^\dagger\chi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

with the covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu(x) \tag{2}$$

and $F^{\mu\nu}$, the electro-magnetic field strength tensor. Use the Feynman gauge fixing term.

In this exercise we are interested in obtaining the time-ordered 4-point correlation function for two fields of type $\phi$ and two fields of type $\chi$. To be precise we want to compute the first interaction term of $\phi$ with $\chi$ in the expansion in the perturbative parameter $e$.

**a)** First find the interaction Hamiltonian $H_{\text{int}}$ of the theory.

**b)** Perform an expansion of the time ordered 4-point correlation function in $e$ up to the first term that allows for an interaction with the gauge field $A_\mu$

$$\begin{aligned}&\langle 0|\mathrm{T}\left\{\phi(x_1)\phi^\dagger(x_2)\chi(x_3)\chi^\dagger(x_4)\right\}|0\rangle_{\text{int}}\\ &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathrm{T}\left\{\phi(x_1)\phi^\dagger(x_2)\chi(x_3)\chi^\dagger(x_4)\exp\left[-i\int_{-T}^{T} dt\, H_{\text{int}}(t)\right]\right\}|0\rangle}{\langle 0|\mathrm{T}\left\{\exp\left[-i\int_{-T}^{T} dt\, H_{\text{int}}(t)\right]\right\}|0\rangle}.\end{aligned} \tag{3}$$

*Hint:* You may discard the terms which do not contribute to the correlator.

**c)** Make use of Wick's theorem to contract the fields in the interaction term you obtained in problem b).

**d)** Focus on the contribution(s) where $\phi$ and $\chi$ interact non-trivially: Insert the Fourier transformed propagators of the scalar and vector fields into your result

$$\begin{aligned}G_{\mathrm{F}}(x-y) &= i\langle 0|\mathrm{T}\{\phi^\dagger(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-y)}}{p^2+m^2-i\epsilon},\\ G_{\mathrm{F}}^{\mu\nu}(x-y) &= i\langle 0|\mathrm{T}\{A_\mu(x)A_\nu(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4}\frac{\eta^{\mu\nu}e^{-ip(x-y)}}{p^2-i\epsilon}.\end{aligned} \tag{4}$$

Simplify your result by performing the integration over the internal spatial variables. How can you interpret the individual factors in your result?

**e)** How can you interpret the terms that do not lead to an interaction of $\phi$ and $\chi$? Can you find a diagrammatic representation of those terms? How do you interpret the limit of $T \to \infty$ in equation (3)?

**f)** *Optional:* How will your result in d) change if you use a different gauge? E.g. use a Lorentz gauge fixing term with $\xi \neq 1$. *Hint:* The gauge affects only $G_{\mathrm{F}}^{\mu\nu}$; try partial fractions to simplify the result.

# Quantum Field Theory I — Problem Set 10

ETH Zurich, HS12 — G. Abelof, J. Cancino, F. Dulat, B. Mistlberger, Prof. N. Beisert

## 1. Linear sigma model with trivial vacuum

Consider a model of $N$ real scalar fields $\Phi^i$ that couple to each other through a quartic interaction that is symmetric under $SO(N)$ rotations of the $N$ fields. The Lagrangian of this model is

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu \Phi^i \partial_\mu \Phi^i - \tfrac{1}{2}m^2 \Phi^i \Phi^i - \tfrac{1}{8}\lambda\left(\Phi^i\Phi^i\right)^2 . \tag{1}$$

**a)** Derive the corresponding Hamiltonian and show that

$$V(\Phi) = \tfrac{1}{2}m^2\Phi^i\Phi^i + \tfrac{1}{8}\lambda\left(\Phi^i\Phi^i\right)^2 \tag{2}$$

is the potential term of the Hamiltonian.

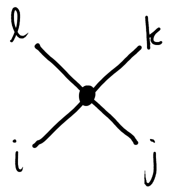
First consider the case $m^2 > 0$, convince yourself that for $\lambda = 0$ the Hamiltonian is just an $N$-fold copy of the Klein–Gordon Hamiltonian. For small $\lambda$ we can calculate a perturbation series in $\lambda$.

**b)** Show that the Wick contraction of the $\Phi^i$ fields is

$$\underbrace{\Phi^i(x)\,\Phi^j}(y) = -i\delta^{ij}G_\text{F}(x-y), \tag{3}$$

where $G_\text{F}$ is the Feynman propagator for a Klein-Gordon scalar field of mass $m$.

**c)** Show that there is one interaction vertex given by




$$= -i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right). \tag{4}$$

**d)** Let $N = 2$ and compute at leading order in $\lambda$ the differential cross section $d\sigma/d\Omega$ for

$$\Phi^1\Phi^1 \to \Phi^2\Phi^2, \tag{5}$$
$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \tag{6}$$
$$\Phi^1\Phi^1 \to \Phi^1\Phi^1. \tag{7}$$

Note that the differential cross section for four particles in the centre of mass frame is given by

$$\frac{d\sigma}{d\Omega} = \frac{|M|^2}{64\pi^2 s}, \tag{8}$$

where $s$ is the centre of mass energy squared of the system and $M$ is the matrix element describing the process.

⟶

## 2. Linear sigma model with non-trivial vacuum

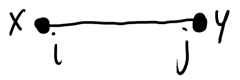
Next we consider the case where $m^2 =: -\mu^2 < 0$. Convince yourself that $V(\Phi)$ has a local maximum at $\Phi^i = 0$. As the potential is bounded from below, the minimum must be located at a non-vanishing value of $\Phi^i$. Moreover, the theory is invariant under global $SO(N)$ rotations of the fields, and all points on the sphere with equal $|\Phi|$ must also be minima of $V(\Phi)$. The ground state of our field theory is therefore given by some non-zero constant field $\Phi^i$. We choose $\Phi^i$ to point along the $N$-th direction or use a $SO(N)$ rotation to that end. We parametrise the quantum fields around the vacuum as

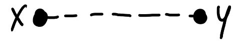
$$\Phi^i(x) = \phi^i(x), \qquad i = 1, \ldots, N-1, \tag{9}$$
$$\Phi^N(x) = v + \sigma(x). \tag{10}$$

**a)** Determine the vacuum expectation value $v$, i.e. the field value in the minimum, in terms of $\mu$ and $\lambda$ by minimising the potential $V(\Phi)$.

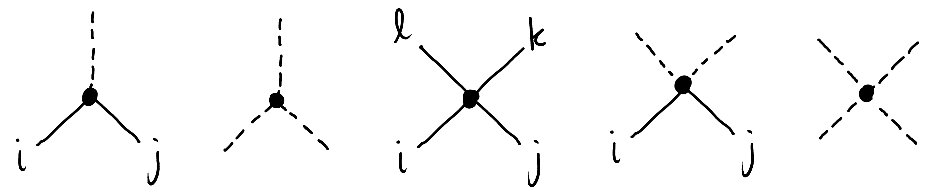
**b)** Insert the ansatz for $\Phi$ in terms of $\phi$ and $\sigma$ and the expression for $v$ into the Lagrangian (or Hamiltonian) and show that the new Lagrangian (Hamiltonian) describes a theory of a massive field $\sigma$ and $N-1$ massless fields $\phi^i$.

**c)** Convince yourself that the $\phi$ and $\sigma$ fields interact through a new set of cubic and quartic vertices and determine the Feynman rules for all propagators and vertices.

$\phi^i(x)\,\phi^j(y) =$ 

$\sigma(x)\,\sigma(y) =$ 

(11)

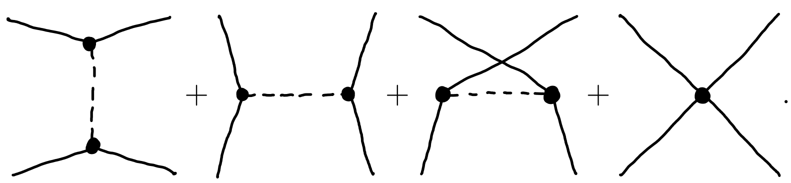


(12)

**d)** Let $N = 2$ and compute at leading order in $\lambda$ the differential cross section $d\sigma/d\Omega$ for

$$\phi^1\phi^1 \to \phi^2\phi^2, \tag{13}$$
$$\phi^1\phi^2 \to \phi^1\phi^2, \tag{14}$$
$$\phi^1\phi^1 \to \phi^1\phi^1. \tag{15}$$

Note that there are now four Feynman diagrams contributing to the amplitude at leading order

$M =$ 
(16)

## 1. Optical theorem

Use the unitarity of the S-matrix, $S^\dagger S = 1$, to show that

$$M_{fi} - M^*_{if} = i \sum_n (2\pi)^4 \delta^4(p_f - p_n) M_{fn} M^*_{in} \,, \tag{1}$$

with $S_{ij} = \delta_{ij} + (2\pi)^4 \delta^4(p_i - p_j) i M_{ij}$.

## 2. Møller scattering

**a)** Calculate the $\mathcal{O}(e^2)$ contribution to the scattering matrix element for Møller scattering:

$$e^-(p_1, \alpha_1) + e^-(p_2, \alpha_2) \longrightarrow e^-(p_3, \alpha_3) + e^-(p_4, \alpha_4) \tag{2}$$

through direct evaluation in position space.

**b)** Repeat the calculation in part a) using the Feynman rules for QED in momentum space.

## 3. Kinematics in 2 → 2 scattering

Consider a $2 \to 2$ particle scattering process with the kinematics $p_1 + p_2 \to p_3 + p_4$.

**a)** Show that in the centre-of-mass frame the energies $e(\vec{p}_i)$ and the norms of momenta $|\vec{p}_i|$ of the incoming and the outgoing particles are entirely fixed by the total centre-of-mass energy $s$ and the particle masses $m_i$.

**b)** Show that the scattering angle $\theta$ between $\vec{p}_1$ and $\vec{p}_3$ is given by

$$\theta = \arccos\left(\frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)}\sqrt{\lambda(s, m_3^2, m_4^2)}}\right), \tag{3}$$

with the Mandelstam variables given by

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2, \qquad u = -(p_1 - p_4)^2, \tag{4}$$

and the Källén function defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{5}$$

**c)** Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

**d)** Determine $t_{\text{min}}$ and $t_{\text{max}}$ from the condition $|\cos\theta| \le 1$, and study the behaviour of $t_{\text{min}}$ and $t_{\text{max}}$ in the limit $s \gg m_i^2$.

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## 4. Muon pair production

Follow the steps below to calculate the total cross section for the process $e^+e^- \to \mu^+\mu^-$.

**a)** Draw all the diagrams that contribute to this process at the lowest non-trivial order, and use the Feynman rules for QED in momentum space to obtain the scattering amplitude $M$.

**b)** Compute $|M|^2$. Assuming that the particle spins are not measured, sum over the spins of the outgoing particle, and average over those of the incoming ones. This should help you bring your expression for $|M|^2$ into a much simpler form. *Hint:* You might find the completeness relations for spinors useful.

**c)** The differential cross section in the center-of-mass frame is given by

$$d\sigma = \frac{|M|^2}{4|\vec{p}_1|\sqrt{s}} \frac{d^3\vec{p}_3}{(2\pi)^3\, 2e(\vec{p}_3)} \frac{d^3\vec{p}_4}{(2\pi)^3\, 2e(\vec{p}_4)} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4). \tag{6}$$

Use the result for $|M|^2$ that you obtained above, and integrate over $\vec{p}_3$ and $\vec{p}_4$ to obtain the total cross section $\sigma = \int d\sigma$.

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## 1. Volume of higher-dimensional spheres

The integrands of $D$-dimensional loop integrals often are spherically symmetric functions $F(\vec{x}) = F(|\vec{x}|)$ (or they can be brought into this form, see Problem 2). The angular part of the integral in spherical coordinates yields the volume of the $(D-1)$-dimensional sphere $S^{D-1}$

$$\int d^D\vec{x}\, F(|\vec{x}|) = \mathrm{Vol}(S^{D-1}) \int_0^\infty r^{D-1} dr\, F(r). \tag{1}$$

In particular, in view of the dimensional regularisation scheme, where $D$ is assumed to be a real number, we need a suitable formula for the volume as an analytic function of $D$. Use the well-known result

$$\int_{-\infty}^{\infty} dx\, \exp(-x^2) = \sqrt{\pi}, \tag{2}$$

to show that the volume of the $(D-1)$-sphere is

$$\mathrm{Vol}(S^{D-1}) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{3}$$

## 2. Feynman and Schwinger parameters

**a)** To evaluate loop diagrams one combines propagators with the use of *Feynman parameters*. The basic version is

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}, \tag{4}$$

but it can be generalised to $n$ propagators elevated to some arbitrary power

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}} = \frac{\Gamma\left(\sum_{i=1}^n \nu_i\right)}{\prod_{i=1}^n \Gamma(\nu_i)} \int_0^1 \left(\prod_{i=1}^n dx_i\right) \delta\left(1 - \sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{\nu_i - 1}}{\left[\sum_{i=1}^n x_i A_i\right]^{\sum_{i=1}^n \nu_i}}. \tag{5}$$

Prove (5) recursively.

**b)** Another useful parametrisation is the *Schwinger parametrisation*:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty d\alpha\, \alpha^{\nu-1} e^{-\alpha A}. \tag{6}$$

Prove (6).

## 3. Electron self energy structure

In QED, the electron two-point function $F(p,q) = -i(2\pi)^4\delta^4(p+q)M(p)$ receives contributions from self energy diagrams.

**a)** Draw the Feynman diagrams corresponding to the one- and two-loop contributions. Which of these diagrams are one-particle irreducible?

**b)** For the one-loop case, write down the expression for $M(p)$ using the massive QED Feynman rules in momentum space and argue why the integral is divergent.

**c)** Explain why one can make the ansatz

$$M = p \cdot \gamma\, M_{\mathrm{V}} + m\, M_{\mathrm{S}}, \tag{7}$$

where $M_{\mathrm{V,S}}$ are scalar functions. Write down integral expressions for them.

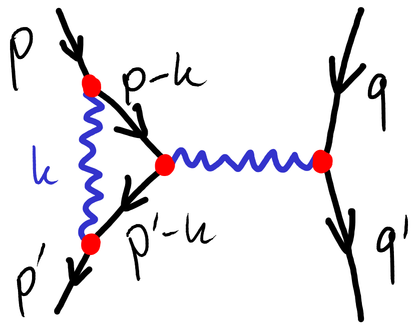
## 1. A one-loop correction to scattering in QED

The aim of this exercise is to gain an insight into the calculation of loop corrections to scattering amplitudes. To this end consider the one-loop corrections to $e^-e^- \to e^-e^-$ scattering in QED.

**a)** Draw all amputated and connected graphs that would contribute to this process. You should find 10 different contributions.

**b)** How does the field strength renormalisation factor for the spinors, $Z_\psi = 1 + Z_\psi^{(2)} + \ldots$, contribute at this perturbative order? How does the field strength renormalisation of the photon $Z_A$ contribute to the process? Can you sketch suitable Feynman graphs?

Now focus on the following diagram:




**c)** Write the scattering matrix element corresponding to the amputated Feynman graph, and bring it to the following form

$$iM = \bar{u}(\vec{q}\,')(-ie\gamma^\mu)u(\vec{q})\,\frac{-i}{(p-p')^2 - i\epsilon}\int \frac{d^Dk}{(2\pi)^D}\,\bar{u}(\vec{p}\,')\,\frac{Z_\mu}{N'}\,u(\vec{p}). \tag{1}$$

**d)** Use a suitable Feynman parametrisation to rewrite the denominator $N'$ as

$$\frac{1}{N'} = 2\int_0^1 dx \int_0^1 dy \int_0^1 dz\,\frac{\delta(1-x-y-z)}{N^3}\,, \tag{2}$$

where

$$N = k^2 - 2k\cdot(xp + yp') - i\epsilon. \tag{3}$$

Complete the square and show that $N$ can be written as

$$N = k'^2 + (1-z)^2m^2 + xy\,(p-p')^2 - i\epsilon, \qquad k' = k - xp - yp'. \tag{4}$$

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**e)** Show that the numerator $Z^\mu$ can be brought to the form

$$Z^\mu = \left(k'^2 + 2(1-4z+z^2)m^2 - 2(z+xy)(p-p')^2\right)\gamma^\mu \\ - z(1-z)m[\gamma^\mu,\gamma^\nu](p'-p)_\nu. \tag{5}$$

To do so, use:

- the anticommutation relations for $\gamma$-matrices

$$(p\cdot\gamma)\gamma^\mu = -2p^\mu - \gamma^\mu(\gamma\cdot p), \tag{6}$$

- the Dirac equation,

$$\bar{u}(\vec{p}\,')\gamma^\mu p'_\mu = m\bar{u}(\vec{p}\,'), \qquad p_\mu\gamma^\mu u(\vec{p}) = mu(\vec{p}), \tag{7}$$

- the symmetry of the integration over $k'$, which allows the following tensorial replacements in the numerator

$$k'^\mu \to 0, \qquad k'^\mu k'^\nu \to \frac{1}{D}\eta^{\mu\nu}k'^2, \tag{8}$$

- the symmetry of the integral under the interchange $x \leftrightarrow y$,
- the Gordon identity

$$\bar{u}(\vec{p}\,')\gamma^\mu u(\vec{p}) = \frac{1}{2m}\,\bar{u}(\vec{p}\,')\left(-(p+p')^\mu - \tfrac{1}{2}[\gamma^\mu,\gamma^\nu](p'-p)_\nu\right)u(\vec{p}). \tag{9}$$

For the remainder of this problem, you may assume that the virtuality of the photon is small, $(p-p')^2 \ll m^2$.

**f)** Using the results obtained in exercise sheet 12, integrate over the loop momentum $k'$. *Note:* Split off a divergent contribution, and cut off the integral as discussed in the lecture. Can you interpret the residual dependence on the cutoff?

**g)** Integrate over $x$ and $y$. *Note:* Cut off the integral if needed. Can you interpret the residual dependence on the cutoff?