Quantum Field Theory I — Problem Set 11

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1. Optical theorem

Use the unitarity of the S-matrix, $S^\dagger S = 1$, to show that

$$M_{fi} - M^*_{if} = i \sum_n (2\pi)^4 \delta^4(p_f - p_n) M_{fn} M^*_{in} \,, \tag{1}$$

with $S_{ij} = \delta_{ij} + (2\pi)^4 \delta^4(p_i - p_j) i M_{ij}$.

2. Møller scattering

a) Calculate the $\mathcal{O}(e^2)$ contribution to the scattering matrix element for Møller scattering:

$$e^-(p_1, \alpha_1) + e^-(p_2, \alpha_2) \longrightarrow e^-(p_3, \alpha_3) + e^-(p_4, \alpha_4) \tag{2}$$

through direct evaluation in position space.

b) Repeat the calculation in part a) using the Feynman rules for QED in momentum space.

3. Kinematics in 2 → 2 scattering

Consider a $2 \to 2$ particle scattering process with the kinematics $p_1 + p_2 \to p_3 + p_4$.

a) Show that in the centre-of-mass frame the energies $e(\vec{p}_i)$ and the norms of momenta $|\vec{p}_i|$ of the incoming and the outgoing particles are entirely fixed by the total centre-of-mass energy s and the particle masses m_i.

b) Show that the scattering angle θ between $\vec{p}_1$ and $\vec{p}_3$ is given by

$$\theta = \arccos\left(\frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\sqrt{\lambda(s, m_1^2, m_2^2)}\sqrt{\lambda(s, m_3^2, m_4^2)}} \right), \tag{3}$$

with the Mandelstam variables given by

$$s = -(p_1 + p_2)^2, \qquad t = -(p_1 - p_3)^2, \qquad u = -(p_1 - p_4)^2, \tag{4}$$

and the Källén function defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{5}$$

c) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.

d) Determine t_{min} and t_{max} from the condition $|\cos\theta| \le 1$, and study the behaviour of t_{min} and t_{max} in the limit $s \gg m_i^2$.

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4. Muon pair production

Follow the steps below to calculate the total cross section for the process $e^+e^- \to \mu^+\mu^-$.

a) Draw all the diagrams that contribute to this process at the lowest non-trivial order, and use the Feynman rules for QED in momentum space to obtain the scattering amplitude M.

b) Compute $|M|^2$. Assuming that the particle spins are not measured, sum over the spins of the outgoing particle, and average over those of the incoming ones. This should help you bring your expression for $|M|^2$ into a much simpler form. *Hint:* You might find the completeness relations for spinors useful.

c) The differential cross section in the center-of-mass frame is given by

$$d\sigma = \frac{|M|^2}{4|\vec{p}_1|\sqrt{s}} \frac{d^3\vec{p}_3}{(2\pi)^3\, 2e(\vec{p}_3)} \frac{d^3\vec{p}_4}{(2\pi)^3\, 2e(\vec{p}_4)} (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4). \tag{6}$$

Use the result for $|M|^2$ that you obtained above, and integrate over $\vec{p}_3$ and $\vec{p}_4$ to obtain the total cross section $\sigma = \int d\sigma$.