Quantum Field Theory I — Problem Set 6

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1. Helicity and Chirality

In four dimensions we can define the chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{1}$$

a) Show that γ^5 satisfies

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \tag{2}$$

$$\left(\gamma^5\right)^2 = 0. \tag{3}$$

b) The helicity operator $h(\vec{p})$ is defined as

$$h(\vec{p}) = \frac{1}{|\vec{p}|}\begin{pmatrix} \sigma^i p_i & 0 \\ 0 & \sigma^i p_i \end{pmatrix}. \tag{4}$$

Show that helicity and chirality are equivalent for a massless spinor $u_s(\vec{p})$.

c) Consider the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi. \tag{5}$$

Find the corresponding Hamiltonian.

d) Show that chirality is not conserved for a massive fermion. *Hint:* You do not need to compute the time-evolution, just show that it is non-trivial.

e) Show that helicity is conserved but not Lorentz invariant.

f) Show that the Dirac Lagrangian is invariant under a chiral transformation $U = \exp(-i\alpha\gamma^5)$ of the fields for $m = 0$, and derive the associated conserved current. Show that having a non-zero mass breaks the symmetry.

2. Discrete symmetries

Recall the $\mathit{\Gamma}^X$ matrices from the last exercise sheet. Using these products of γ matrices, we can define the following bilinears:

$$S = \bar{\psi}\mathit{\Gamma}^S\psi, \tag{6}$$

$$P = \bar{\psi}\mathit{\Gamma}^P\psi, \tag{7}$$

$$V^\mu = \bar{\psi}\mathit{\Gamma}^{V,\mu}\psi, \tag{8}$$

$$A^\mu = \bar{\psi}\mathit{\Gamma}^{A,\mu}\psi, \tag{9}$$

$$T^{\mu\nu} = \bar{\psi}\mathit{\Gamma}^{T,\mu\nu}\psi. \tag{10}$$

a) Calculate their behaviour under parity transformations $P(\psi(t,\vec{x})) = \gamma^0\psi(t,-\vec{x})$.

b) Show that the Dirac Lagrangian (5) is invariant under CPT as well as under P.

c) Starting from the Dirac Lagrangian, write down a similar Lagrangian that is CPT invariant but not P invariant.

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3. Electrodynamics

Consider the Lagrange density

$$\mathcal{L}(A_\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\,, \qquad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{11}$$

and J^μ is some external source field.

a) Show that the Euler–Lagrange equations are the inhomogenous Maxwell equations. The usual electromagnetic fields are defined by $E^i = -F^{0i}$ and $\epsilon^{ijk}B^k = -F^{ij}$. Are these all Maxwell equations?

b) Construct the stress-energy tensor for this theory.

c) Convince yourself that the stress-energy tensor is not symmetric. In order to make it symmetric consider

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu,\nu}\,, \tag{12}$$

where $K^{\lambda\mu\nu}$ is anti-symmetric in the first two indices. By taking

$$K^{\lambda\mu,\nu} = F^{\mu\lambda}A^\nu \tag{13}$$

show that the modified stress energy tensor $\hat{T}^{\mu\nu}$ is symmetric, and that it leads to the standard formulae for the electromagnetic energy and momentum densities

$$\mathcal{E} = \tfrac{1}{2}(\vec{E}^{\,2} + \vec{B}^{\,2})\,, \qquad \vec{\mathcal{S}} = \vec{E} \times \vec{B}\,. \tag{14}$$

d) *For fun:* Show that all of Maxwell's equations can summarised as

$$\gamma^\nu\gamma^\rho\gamma^\sigma\partial_\nu F_{\rho\sigma} = -2\gamma^\nu J_\nu. \tag{15}$$