Quantum Field Theory I — Problem Set 4

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1. Properties of γ-matrices

The γ-matrices satisfy a *Clifford algebra*,[1]

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}\mathbf{1}. \tag{1}$$

a) Show the following contraction identities using (1):

1. $\gamma^\mu\gamma_\mu = -4 \cdot \mathbf{1}$.
2. $\gamma^\mu\gamma^\nu\gamma_\mu = 2\gamma^\nu$.
3. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}\mathbf{1}$.
4. $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = 2\gamma^\sigma\gamma^\rho\gamma^\nu$.

b) Show the following trace properties using (1):

1. $\operatorname{tr}\gamma^{\mu_1}\cdots\gamma^{\mu_n} = 0$ if n is odd.
2. $\operatorname{tr}\gamma^\mu\gamma^\nu = -4\eta^{\mu\nu}$.
3. $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$.

2. Dirac and Weyl representations of the γ-matrices

Using the Pauli matrices together with the identity,

$$\sigma^0 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

we can realize the *Dirac representation* of the γ-matrices,

$$\gamma^0_{\mathrm{D}} \equiv \sigma^0 \otimes \sigma^3, \qquad \gamma^j_{\mathrm{D}} \equiv \sigma^j \otimes i\sigma^2 \quad (j = 1, 2, 3), \tag{3}$$

where

$$A \otimes B \equiv \begin{pmatrix} b_{11}A & b_{12}A \\ b_{21}A & b_{22}A \end{pmatrix}. \tag{4}$$

Denoting the Pauli matrices collectively by σ^μ and defining $(\bar\sigma^0, \bar\sigma^i) = (\sigma^0, -\sigma^i)$. we can then define the γ-matrices in the *Weyl representation*:

$$\gamma^\mu_{\mathrm{W}} \equiv \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}. \tag{5}$$

Show that both representations satisfy the Clifford algebra (1). Can you show their equivalence, i.e. $\gamma^\mu_{\mathrm{W}} = T\gamma^\mu_{\mathrm{D}}T^{-1}$ for some matrix T?

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[1] The minus sign is due to our choice of metric $\eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$! Alternatively, we might use a plus sign (as in the opposite signature) and instead multiply all γ-matrices by a factor of i.

3. Spinors, spin sums and completeness relations

In this exercise we will use the Weyl representation (5) defined in the previous exercise.

a) Show that $(p \cdot \sigma)(p \cdot \bar{\sigma}) = -p^2$.

b) Prove that the below 4-spinor $u_s(\vec{p})$ solves Dirac's equation $(p_\mu \gamma^\mu - m\mathbf{1})u_s(\vec{p}) = 0$

$$u_s(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi_s \\ \sqrt{p \cdot \bar{\sigma}}\, \xi_s \end{pmatrix}, \tag{6}$$

where $\xi_\pm$ form a basis of 2-spinors.

c) Suppose, the 2-spinors ξ_+ and ξ_- are orthonormal. What does it imply for $\xi_s^\dagger \xi_s$ and

$$\sum_{s \in \{+,-\}} \xi_s \xi_s^\dagger \,? \tag{7}$$

d) Show that $\bar{u}_s(\vec{p}) u_s(\vec{p}) = 2m$ for $s \in \{+,-\}$.

e) Show the *completeness relation*:

$$\sum_{s \in \{+,-\}} u_s(\vec{p}) \bar{u}_s(\vec{p}) = p_\mu \gamma^\mu + m\mathbf{1}. \tag{8}$$

4. Gordon identity

Prove the *Gordon identity*,

$$\bar{u}_t(\vec{q}) \gamma^\mu u_s(\vec{p}) = \frac{1}{2m}\, \bar{u}_t(\vec{q}) \big[-(q+p)^\mu - \tfrac{1}{2}[\gamma^\mu, \gamma^\nu](q-p)_\nu\big] u_s(\vec{p}). \tag{9}$$

Hint: You can do this using just (1).