Quantum Field Theory I — Problem Set 2

ETH Zurich, HS12 — G. Abelof, J. Cancino, F. Dulat, B. Mistlberger, Prof. N. Beisert

1. Causality

Consider a scalar field $\phi(x)$ as defined in the lecture. First we want to calculate the amplitude

$$\Delta_+(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle \tag{1}$$

for a particle to propagate from point x to point y.

a) Calculate $\Delta_+(x-y)$ for time-like separation, i.e. $x^0 - y^0 = t$, $x^i - y^i = 0$.

b) Calculate $\Delta_+(x-y)$ for space-like separation, i.e. $x^0 - y^0 = 0$, $x^i - y^i = r^i$.

The next thing we need to check is whether a measurement at x can affect another measurement at y. To do this one computes the commutator $[\phi(x), \phi(y)]$. If it vanishes, the two measurements cannot affect each other and causality is preserved.

c) Show that the commutator vanishes for a space-like separation of x and y.

2. Complex scalar field

We want to investigate the theory of a complex scalar field $\phi = \phi(x)$. The theory is described by the Lagrangian (density):

$$\mathcal{L} = -\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \tag{2}$$

As a complex scalar field has two degrees of freedom, we can treat ϕ and ϕ^* as independent fields with one degree of freedom each.

a) Find the conjugate momenta $\pi(\vec{x})$ and $\pi^*(\vec{x})$ to $\phi(\vec{x})$ and $\phi^*(\vec{x})$ and the canonical commutation relations. (Note: we choose $\pi = \partial\mathcal{L}/\partial\dot{\phi}$ rather than $\pi = \partial\mathcal{L}/\partial\dot{\phi}^*$.)

b) Find the Hamiltonian of the theory.

c) Introduce creation and annihilation operators to diagonalise the Hamiltonian.

d) Show that the theory contains two sets of particles of mass m.

e) Consider the conserved charge

$$Q = -\frac{i}{2}\int d^3\vec{x}\,(\pi\phi - \phi^*\pi^*). \tag{3}$$

Rewrite it in terms of ladder operators and determine the charges of the two particle species.

3. Momentum

Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{4}$$

of a real scalar field $\phi = \phi(x)$.

a) Write down the stress-energy tensor of the theory using the general result obtained in the previous exercise sheet.

b) Derive

$$P^\mu = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2e(\vec{p})}\, p^\mu(\vec{p})\, a^\dagger(\vec{p}) a(\vec{p}) \tag{5}$$

starting from $P^\mu = \int d^3\vec{x}\, T^{0\mu}$.

c) Calculate the commutator $[P^\mu, \phi(x)]$ and interpret the result.