2 Classical Free Scalar Field

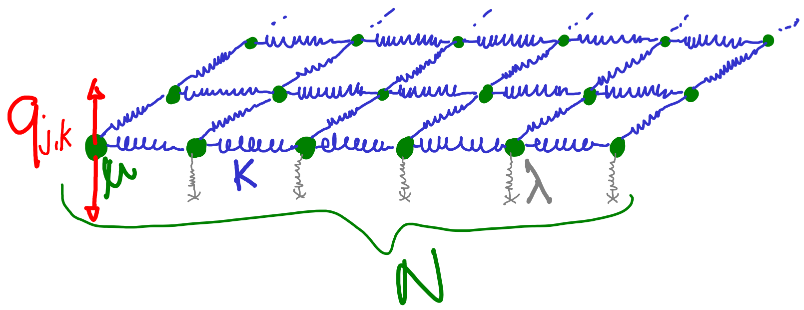
In the following we shall discuss one of the simplest field theory models, the classical non-interacting relativistic scalar field.

2.1 Spring Lattice

Before considering field, start with an approximation we can certainly handle: lattice.

Consider an atomic lattice:

- 1D or 2D cubic lattice,
- atoms are coupled to neighbours by springs,[1]
- atoms are coupled to rest position by springs,
- atoms can move only orthogonally to lattice (transverse),
- boundaries: periodic identification.



(2.1)

model parameters and variables:

- lattice separation r,
- number of atoms N (in each direction),
- mass of each atom μ,
- lattice spring constant κ,
- return spring constant λ,
- shift orthogonal to lattice $q_{j,k}$

Lagrangian Formulation. Lagrange function

$$L = L_{\text{kin}} - V_{\text{lat}} - V_{\text{rest}} \tag{2.2}$$

[1] Springs are useful approximations because they model first deviation from rest position; always applies to small excitations.

Standard non-relativistic kinetic terms

$$L_{\text{kin}} = \tfrac{1}{2}\mu \sum_{i,j=1}^{N} \dot{q}_{i,j}^2. \tag{2.3}$$

Potential for springs between atoms (ignore x, y-potential)[2]

$$V_{\text{lat}} = \tfrac{1}{2}\kappa \sum_{i,j=1}^{N} (q_{i-1,j} - q_{i,j})^2 + \tfrac{1}{2}\kappa \sum_{i,j=1}^{N} (q_{i,j-1} - q_{i,j})^2. \tag{2.4}$$

Some spring potential to drive atoms back to rest position

$$V_{\text{rest}} = \tfrac{1}{2}\lambda \sum_{i,j=1}^{N} q_{i,j}^2. \tag{2.5}$$

Quadratic in q's: bunch of coupled HO's. Equations of motion

$$\mu \ddot{q}_{i,j} \begin{array}{l} - \kappa(q_{i-1,j} - 2q_{i,j} + q_{i+1,j}) \\ - \kappa(q_{i,j-1} - 2q_{i,j} + q_{i,j+1}) \end{array} + \lambda q_{i,j} = 0. \tag{2.6}$$

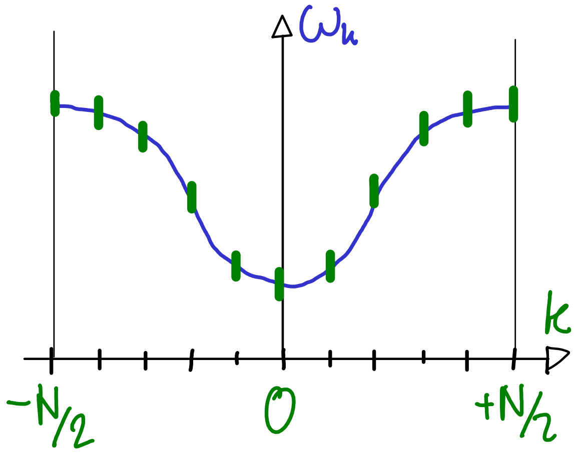
Note: spatially homogeneous equations. Use discrete Fourier transform to solve (respect periodicity)

$$\begin{aligned} q_{i,j}(t) = {} & \frac{1}{N^2} \sum_{k,l=1}^{N} \frac{c_{k,l}}{\sqrt{2\mu\omega_{k,l}}} \exp\left(-\frac{2\pi i}{N}(ki + lj) - i\omega_{k,l} t\right) \\ & + \frac{1}{N^2} \sum_{k,l=1}^{N} \frac{c_{k,l}^*}{\sqrt{2\mu\omega_{k,l}}} \exp\left(\frac{2\pi i}{N}(ki + lj) + i\omega_{k,l} t\right) \end{aligned} \tag{2.7}$$

Used freedom to define coefficients $c_{k,l}$ to introduce prefactors.Complex conjugate coefficients $c_{k,l}^*$ in second term ensure reality of $q_{i,j}$. Note: $c_{k,l}^*$ represents mode of opposite momentum and energy w.r.t. $c_{k,l}$!

E.o.m. translates to dispersion relation:

$$\mu\omega_{k,l}^2 = \lambda + 4\kappa \sin^2 \frac{\pi k}{N} + 4\kappa \sin^2 \frac{\pi l}{N}\,. \tag{2.8}$$



(2.9)

[2]Can ignore due to $d^2 = d_x^2 + d_y^2 + d_z^2$, shifts potential by an irrelevant constant. Moreover longitudinal and transverse excitations decouple.

Hamiltonian Formulation. Define momenta

$$p_{i,j} := \frac{\partial L}{\partial \dot{q}_{i,j}} = \mu \dot{q}_{i,j}. \tag{2.10}$$

Derive Hamiltonian function as Legendre transform of L

$$H = H_{\text{kin}} + V_{\text{lat}} + V_{\text{rest}} \quad \text{with} \quad H_{\text{kin}} = \frac{1}{2\mu} \sum_{i,j=1}^{N} p_{i,j}^2. \tag{2.11}$$

Then define canonical Poisson brackets

$$\{f, g\} := \sum_{i,j=1}^{N} \left(\frac{\partial f}{\partial q_{i,j}} \frac{\partial g}{\partial p_{i,j}} - \frac{\partial f}{\partial p_{i,j}} \frac{\partial g}{\partial q_{i,j}} \right). \tag{2.12}$$

In other words $\{q_{i,j}, p_{j,k}\} = \delta_{i,k}\delta_{j,l}$ and $\{q, q\} = \{p, p\} = 0$.

Fourier Modes. Introduce new complex variables (Fourier transform)

$$a_{k,l} = \frac{1}{\sqrt{2\mu\omega_{k,l}}} \sum_{i,j=1}^{N} \exp\left(\frac{2\pi i}{N}(ki + lj) \right) \big(\mu\omega_{k,l} q_{i,j} + i p_{i,j} \big). \tag{2.13}$$

Transformed Hamiltonian is very simple

$$H = \frac{1}{N^2} \sum_{k,l=1}^{N} \omega_{k,l} a_{k,l}^* a_{k,l}. \tag{2.14}$$

Poisson brackets for new variables are simple, too

$$\begin{aligned} \{a_{i,j}, a_{k,l}^*\} &= -i\, N^2\, \delta_{i,k}\delta_{j,l}, \\ \{a_{i,j}, a_{k,l}\} &= \{a_{i,j}^*, a_{k,l}^*\} = 0. \end{aligned} \tag{2.15}$$

Can convince oneself that equations of motion hold

$$\begin{aligned} \dot{a}_{k,l} &= -\{H, a_{k,l}\} = -i\omega_{k,l} a_{k,l}, \\ \dot{a}_{k,l}^* &= -\{H, a_{k,l}^*\} = +i\omega_{k,l} a_{k,l}^*. \end{aligned} \tag{2.16}$$

and solved by above Lagrangian solution

$$a_{k,l}(t) = c_{k,l} \exp(-i\omega_{k,l} t), \qquad a_{k,l}^*(t) = c_{k,l}^* \exp(+i\omega_{k,l} t). \tag{2.17}$$

2.2 Continuum Limit

Now turn this spring lattice into a smooth field φ:

- send number of sites $N \to \infty$.
- box of size L in all directions; lattice separation $r = L/N \to 0$.
- positions $x = ir = iL/N$,
- field $q_{i,\ldots} = \varphi(\vec{x})$,
- generalise to d spatial dimensions, e.g. $d = 1, 2, 3$.

Some useful rules

$$\sum_{i=1}^{N} \to \frac{1}{r} \int dx, \qquad q_i - q_{i-1} \to r(\partial\varphi) \tag{2.18}$$

Lagrangian Formulation. Substitute this in Lagrangian

$$L \to \int d^d\vec{x} \left(\frac{\mu}{2r^d}\dot{\varphi}^2 - \frac{\kappa}{2r^{d-2}}(\vec{\partial}\varphi)^2 - \frac{\lambda}{2r^d}\varphi^2 \right). \tag{2.19}$$

Diverges as $r \to 0$, but can rescale parameters. Suitable rescalings

$$\mu = r^d\bar{\mu}, \quad \kappa = r^{d-2}\bar{\kappa}, \quad \lambda = r^d\bar{\lambda}, \tag{2.20}$$

Parameters become densities. Lagrangian functional

$$L[\varphi,\dot{\varphi}](t) = \int d^d\vec{x} \left(\tfrac{1}{2}\bar{\mu}\dot{\varphi}^2 - \tfrac{1}{2}\bar{\kappa}(\vec{\partial}\varphi)^2 - \tfrac{1}{2}\bar{\lambda}\varphi^2 \right). \tag{2.21}$$

Can furthermore rescale field $\varphi = \bar{\kappa}^{-1/2}\phi$

$$L[\phi,\dot{\phi}](t) = \int d^d\vec{x} \left(\tfrac{1}{2}\bar{\mu}\bar{\kappa}^{-1}\dot{\phi}^2 - \tfrac{1}{2}(\vec{\partial}\phi)^2 - \tfrac{1}{2}\bar{\lambda}\bar{\kappa}^{-1}\phi^2 \right). \tag{2.22}$$

Derive e.o.m.: start with action functional $S[\phi]$

$$S[\phi] = \int dt\, L[\phi](t) = \int dt\, d^d\vec{x}\, \mathcal{L}\big(\phi(\vec{x},t), \partial_i\phi(\vec{x},t), \dot{\phi}(\vec{x},t)\big); \tag{2.23}$$

useful to express (homogeneous) Lagrange functional $L[\phi](t)$ through Lagrangian density $\mathcal{L}(\phi,\vec{\partial}\phi,\dot{\phi})$ (the Lagrangian).

Vary action functional (discard boundary terms)

$$\delta S[\phi] = \int dt\, d^d\vec{x}\, \delta\phi \left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} \right) + \ldots \overset{!}{=} 0. \tag{2.24}$$

Write general Euler–Lagrange equation for fields

$$\frac{\partial\mathcal{L}}{\partial\phi}(\vec{x},t) - \frac{\partial}{\partial x^i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}(\vec{x},t) - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}(\vec{x},t) = 0. \tag{2.25}$$

In our case

$$-\bar{\mu}\bar{\kappa}^{-1}\ddot{\phi} + \vec{\partial}^2\phi - \bar{\lambda}\bar{\kappa}^{-1}\phi = 0; \tag{2.26}$$

agrees with continuum limit of discrete e.o.m.. Now denote as speed of light c and mass m

$$\bar{\mu}\bar{\kappa}^{-1} = c^{-2} = 1, \qquad \bar{\lambda}\bar{\kappa}^{-1} = m^2, \tag{2.27}$$

to discover Klein–Gordon equation (set $c = 1$)

$$-c^{-2}\ddot{\phi} + \vec{\partial}^2\phi - m^2\phi = 0. \tag{2.28}$$

Plane Wave Solutions. Consider solutions on infinite space and time. Homogeneous equation solved by Fourier transformation

$$\phi(\vec{x},t) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, \alpha(\vec{p}) \exp\bigl(-i\vec{p}{\cdot}\vec{x} - ie(\vec{p})t\bigr) + \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, \alpha^*(\vec{p}) \exp\bigl(+i\vec{p}{\cdot}\vec{x} + ie(\vec{p})t\bigr) \tag{2.29}$$

with (positive) energy $e(\vec{p})$ on mass shell (often called $\omega(\vec{p})$)

$$e(\vec{p}) = \sqrt{\vec{p}^{\,2} + m^2}. \tag{2.30}$$

Agrees with discrete solution identifying momenta as

$$p = 2\pi k/L, \qquad \sum_{k=1}^{N} \to \frac{L}{2\pi}\int dp, \qquad c_{k,\ldots} = \frac{\alpha(\vec{p})}{\sqrt{2e(\vec{p})r^d}}\,. \tag{2.31}$$

Some remarks on factors and conventions:

- Fourier transforms on $\mathbb{R}$ produce factors of 2π, need to be put somewhere.
  Convention to associate $(2\pi)^{-1}$ to every dp: $đp := dp/2\pi$.
  No factors of 2π for dx. No factors of 2π in exponent.
- Combination $d^d\vec{p}/2e(\vec{p})$ is a useful combination: Relativistic covariance. Reason for conversion factor in $c_{k,\ldots}$.

2.3 Relativistic Covariance

The Klein–Gordon equation can be written manifestly relativistically[3] [4]

$$-\partial^\mu\partial_\mu\phi + m^2\phi = -\partial^2\phi + m^2\phi = 0. \tag{2.32}$$

Also Lagrangian and action manifestly relativistic

$$\mathcal{L} = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}m^2\phi^2, \qquad S = \int d^Dx\, \mathcal{L}. \tag{2.33}$$

To understand the relativistic behaviour of the solution, consider integration over a mass shell $p^2 + m^2 = 0$

$$\begin{aligned} &\int d^Dp\, \delta(p^2+m^2)\theta(p^0)\, f(p) \\ &= \int d^d\vec{p}\, de\, \delta(-e^2 + \vec{p}^{\,2} + m^2)\theta(e)\, f(e,\vec{p}) \\ &= \int \frac{d^d\vec{p}\, de}{2e}\, \delta(e - \sqrt{\vec{p}^{\,2}+m^2})\theta(e)\, f(e,\vec{p}) \\ &= \int \frac{d^d\vec{p}}{2e(\vec{p})}\, f(e(\vec{p}),\vec{p}) \end{aligned} \tag{2.34}$$

[3] ∂^2 often written as D'Alembertian $\Box$.
[4] Signature of spacetime is $-{+}{+}{+}$!

Solution is just Fourier transform

$$\phi(x) = \int \frac{d^D p}{(2\pi)^D}\, \phi(p) \exp\bigl(-ip_\mu x^\mu\bigr) \tag{2.35}$$

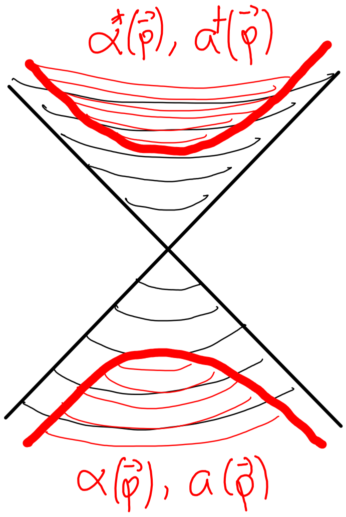
with momentum space field defined on shell only

$$\phi(p) = 2\pi\delta(p^2 + m^2)\bigl(\theta(p_0)\alpha(\vec{p}) + \theta(-p_0)\alpha^*(-\vec{p})\bigr). \tag{2.36}$$

Momentum space e.o.m. obviously satisfied

$$(p^2 + m^2)\phi(p) = 0. \tag{2.37}$$

$\alpha(\vec{p})$ and $\alpha^*(\vec{p})$ define amplitudes on forward/backward mass shells.



(2.38)

Note that $\phi(p)$ obeys reality condition (from $\phi(x)^* = \phi(x)$)

$$\phi(p)^* = \phi(-p). \tag{2.39}$$

2.4 Hamiltonian Field Theory

Now that we have a nice relativistic formulation for the Klein–Gordon field $\phi(x)$, let's separate space from time. [5]

Position Space. Define momentum π (field) conjugate to ϕ

$$\pi(\vec{x}, t) = \frac{\delta L}{\delta\dot{\phi}(\vec{x})}(t) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}}(\vec{x}, t) = \dot{\phi}(\vec{x}, t). \tag{2.40}$$

Determine Hamiltonian

$$\begin{aligned} H[\phi, \pi] &= \int d^d\vec{x}\, \pi\dot{\varphi} - L[\phi, \dot{\phi}] \\ &= \int d^d\vec{x}\, \Bigl(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\partial}\phi)^2 + \tfrac{1}{2}m^2\phi^2\Bigr). \end{aligned} \tag{2.41}$$

[5]Formalism breaks relativistic invariance, physics remains relativistic.

Not relativistically covariant, not designed to be.[6]

Define Poisson brackets for phase space functionals f, g

$$\{f, g\} = \int d^d\vec{x} \left(\frac{\delta f}{\delta\phi(\vec{x})} \frac{\delta g}{\delta\pi(\vec{x})} - \frac{\delta f}{\delta\pi(\vec{x})} \frac{\delta g}{\delta\phi(\vec{x})} \right). \tag{2.42}$$

Poisson brackets of fields yield delta-functions[7]

$$\{\phi(\vec{x}), \pi(\vec{y})\} = \int d^d\vec{z}\, \delta^d(\vec{x}-\vec{z})\delta^d(\vec{y}-\vec{z}) = \delta^d(\vec{x}-\vec{z}). \tag{2.43}$$

Momentum Space. Now introduce momentum modes[8]

$$a(\vec{p}) = \int d^d\vec{x} \exp(i\vec{p}\cdot\vec{x}) \big(e(\vec{p})\phi(\vec{x}) + i\pi(\vec{x})\big), \tag{2.44}$$

and inverse Fourier transformation

$$\begin{aligned} \phi(\vec{x}) = &\int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a(\vec{p}) \exp(-i\vec{p}\cdot\vec{x}) \\ &+ \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, a^*(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}), \\ \pi(\vec{x}) = &-\frac{i}{2} \int \frac{d^d\vec{p}}{(2\pi)^d}\, a(\vec{p}) \exp(-i\vec{p}\cdot\vec{x}) \\ &+ \frac{i}{2} \int \frac{d^d\vec{p}}{(2\pi)^d}\, a^*(\vec{p}) \exp(+i\vec{p}\cdot\vec{x}). \end{aligned} \tag{2.45}$$

Compute Poisson brackets for Fourier modes[9] [10]

$$\{a(\vec{p}), a^*(\vec{q})\} = -i\, 2e(\vec{p})\, (2\pi)^d \delta^d(\vec{p}-\vec{q}). \tag{2.46}$$

In other words

$$\{f, g\} = -i(2\pi)^d \int d^d\vec{p}\; 2e(\vec{p}) \left(\frac{\delta f}{\delta a(\vec{p})} \frac{\delta g}{\delta a^*(\vec{p})} - \frac{\delta f}{\delta a^*(\vec{p})} \frac{\delta g}{\delta a(\vec{p})} \right). \tag{2.47}$$

Hamiltonian translates to

$$H = \frac{1}{2} \int \frac{d^d\vec{p}}{(2\pi)^d}\, a^*(\vec{p}) a(\vec{p}) = \int \frac{d^d\vec{p}}{(2\pi)^d\, 2e(\vec{p})}\, e(\vec{p}) a^*(\vec{p}) a(\vec{p}). \tag{2.48}$$

Hence e.o.m. for field oscillator

$$\begin{aligned} \dot{a}(\vec{p}) &= -\{H, a(\vec{p})\} = -ie(\vec{p})a(\vec{q}), \\ \dot{a}^*(\vec{p}) &= -\{H, a^*(\vec{p})\} = +ie(\vec{p})a^*(\vec{q}). \end{aligned} \tag{2.49}$$

One HO for every momentum. Solution

$$a(\vec{p}, t) = \alpha(\vec{p}) \exp(-ie(\vec{p})t), \quad a^*(\vec{p}, t) = \alpha^*(\vec{p}) \exp(+ie(\vec{p})t). \tag{2.50}$$

---

[6]Hamiltonian governs *time* translation.

[7]Formula for variation $\delta\phi(\vec{x})/\delta\phi(\vec{z}) = \delta^d(\vec{x}-\vec{z})$.

[8]Have some additional factors compared to some literature.

[9]Conventional 2π for delta-function in momentum space.

[10]Factor $2e(\vec{p})$ appropriate relativistic measure for mass shell.