Sheet X

Due: week of December 17

We look at the retarded solution of the scalar wave equation

$$\Box\phi = -4\pi\rho. \tag{1}$$

In the following, a limit of the form

$$\lim_{(v,\xi);\, r\to\infty} \tag{2}$$

will mean taking the limit as $r \to \infty$ along a generator of the past light cone C_v^-, the generator being given by $\xi \in S^2$. We assume that

$$\max_{\Sigma_t}\left|\frac{\partial\rho}{\partial t}\right| \leq \Lambda(t), \qquad \int_{-\infty}^{t_0} \Lambda^2(t)dt < \infty \tag{3}$$

for any t_0, where $\Lambda(t)$ is a positive, non-decreasing function. Furthermore we assume that

$$\mathrm{supp}\rho\big|_{\Sigma_t} \subset B_R \tag{4}$$

for a fixed radius R independent of t.

Question 1 [*Incoming radiation*]: Show that there is no incoming radiation, i.e. show that the incoming flux given by

$$\underline{F}(v) = \lim_{r\to\infty}\int_{S_{v-r,r}} \frac{1}{2}(T_{\underline{L}L} + T_{LL})d\mu_\gamma \tag{5}$$

vanishes, where (see lecture)

$$T_{\underline{L}L} = |\not{d}\phi|^2, \qquad T_{LL} = (L\phi)^2. \tag{6}$$

Hints:

(i) Using arguments similar to the ones used for the limit along a generator of the future null cone (see lecture), show that

$$\lim_{(v,\xi);\, r\to\infty} r\phi = \lim_{u\to-\infty}\int_{P(u,\xi)} \rho, \tag{7}$$

where $P(u,\xi)$ is the null hyperplane as defined on sheet IX.

(ii) Show using (7) that

$$\lim_{(v,\xi);\, r\to\infty} r|\not{d}\phi| = 0. \tag{8}$$

From this deduce that there is no contribution from the first term in (5).

(iii) Use the assumptions on the source ρ to show that

$$\left| \int_{P(u,\xi)} \frac{\partial \rho}{\partial t} \right| \leq \frac{4\pi}{3} R^3 \Lambda(u+R). \tag{9}$$

From this deduce that

$$\lim_{u \to -\infty} \int_{P(u,\xi)} \frac{\partial \rho}{\partial t} = 0. \tag{10}$$

(iv) Use (7) and (10) to show that

$$\lim_{(v,\xi);\, r \to \infty} r L \phi = 0. \tag{11}$$

From this deduce that there is no contribution from the second term in (5).

Question 2 [*Radiated energy from infinite past*]: Show that the amount of energy radiated from the infinite past to any given retarded time is finite.

Hint: The amount of energy radiated from the infinite past to the retarded time u_0 is given by

$$\int_{-\infty}^{u_0} G(u) du, \tag{12}$$

where (see lecture)

$$G(u) = 2 \int_{S^2} \left(\frac{\partial \Phi}{\partial u} \right)^2 (u, \xi) d\mu_{\overset{\circ}{\gamma}}. \tag{13}$$

Use the definition of $\Phi(u, \xi)$ as given on sheet IX together with the assumptions on the source ρ to estimate (12).