# Sheet VI

Due: week of November 5

We consider a perfect fluid in Minkowski spacetime. First we recall the setting from last week's lecture. The equations of motion of the fluid are the conservation laws

$$\nabla_\mu T^{\mu\nu} = 0, \tag{1}$$

$$\nabla_\mu I^\mu = 0, \tag{2}$$

where $T^{\mu\nu}$ is the energy momentum stress tensor of a perfect fluid

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\left((g^{-1})^{\mu\nu} + u^\mu u^\nu\right) \tag{3}$$

and $I^\mu$ is the particle current

$$I^\mu = n u^\mu. \tag{4}$$

Here $\rho$ is the energy per unit volume, $u^\mu$ is the fluid four-velocity (a future-directed unit time-like vectorfield), $p$ denotes the pressure and $n$ is the number of particles per unit volume. In addition we have the equation of state which expresses the energy per particle $E$ as a function of the volume per particle $V$ and the entropy per particle $S$

$$E = E(V,S). \tag{5}$$

According to the first law of thermodynamics the pressure $p$ and the temperature $\theta$ are then given by

$$dE = -pdV + \theta dS. \tag{6}$$

We also have the relations

$$\rho = \frac{E}{V}, \qquad n = \frac{1}{V}. \tag{7}$$

For the following questions use rectangular coordinates.

**Question 1** [*Nonrelativistic limit of conservation laws* ]: Recall that

$$E = Mc^2 + U, \tag{8}$$

where $M$ is the particle mass and $U$ is the internal energy per particle.

(i) Derive in the nonrelativistic limit the equation of the conservation of mass

$$\frac{\partial \mu}{\partial t} + \frac{\partial(\mu v^i)}{\partial x^i} = 0, \tag{9}$$

where $\mu = M/V$ is the mass density and $v$ is the three-velocity according to

$$u^0 = \frac{1}{\sqrt{1-|v|^2/c^2}}, \qquad u^i = \frac{v^i/c}{\sqrt{1-|v|^2/c^2}}. \tag{10}$$

(ii) Derive in the nonrelativistic limit the equation of the conservation of energy

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot f = 0, \tag{11}$$

where $\varepsilon$ is the total energy density given by

$$\varepsilon = \frac{1}{2}\mu |v|^2 + u. \tag{12}$$

Here $u = U/V$ is the internal energy per unit volume. The three-vector $f$ is the total energy flux given by

$$f^i = (\varepsilon + p)v^i. \tag{13}$$

Hint: Use the following combination of (1) and (2)

$$\nabla_\mu \left( T^{0\mu} - Mc^2 I^\mu \right) = 0. \tag{14}$$

**Question 2** [*Adiabatic condition*]: Use (2) to show that the $u$-component of (1), i.e.

$$u_\mu \nabla_\nu T^{\mu\nu} = 0, \tag{15}$$

is the adiabatic condition

$$u^\mu \partial_\mu S = 0. \tag{16}$$

**Question 3** [*Pressureless fluid*]: Show that if $p = 0$ the integral curves of $u$ are timelike geodesics.