Sheet III

Due: week of October 15

Let (x^0, x^1, x^2, x^3) be cylindrical normal coordinates as introduced in the lecture. Here x^0 is the signed arc length from the origin along Γ_0, where Γ_0 is the timelike geodesic of the freely falling reference particle to which the coordinate system is assigned.

Question 1 [*Expansion of the metric to first order*]:

(i) Show that the metric along Γ_0 is equal to the Minkowski metric, i.e.

$$g_{\mu\nu}(x^0, 0, 0, 0) = \eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{1}$$

Hint: Recall from the lecture that at any point $p \in \Gamma_0$

$$\frac{\partial}{\partial x^0} = T, \qquad \frac{\partial}{\partial x^i} = E_i, \tag{2}$$

where T is the tangent vector to Γ_0 at p and (E_1, E_2, E_3) is an orthonormal basis of the local simultaneous space Σ_p.

(ii) Let r be the distance from Γ_0, i.e.

$$r := \sqrt{\sum_{i=1}^{3} (x^i)^2}. \tag{3}$$

Show that to first order in r, the metric is equal to the Minkowski metric, i.e.

$$g_{\mu\nu}(x^0, x^1, x^2, x^3) = \eta_{\mu\nu} + O(r^2). \tag{4}$$

Hints: The metric is covariantly constant, i.e. $\nabla_\mu g_{\nu\lambda} = 0$. The problem reduces therefore to show that $\Gamma^\mu_{\nu\lambda} = 0$ along Γ_0. This can be done using the following three considerations.

(a) Use the fact that the curve

$$\tau \mapsto (\tau, 0, 0, 0) \tag{5}$$

is a geodesic (namely Γ_0), i.e. satisfies the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\sigma} \frac{dx^\lambda}{d\tau} \frac{dx^\sigma}{d\tau} = 0 \tag{6}$$

and deduce that

$$\forall x^0 : \quad \Gamma^\mu_{00}(x^0, 0, 0, 0) = 0. \tag{7}$$

(b) Use the fact that for any three-vector k and any x^0 the curve

$$\tau \mapsto (x^0, k^1\tau, k^2\tau, k^3\tau) \tag{8}$$

is a geodesic, i.e. satisfies (6), and deduce that

$$\forall x^0, \tau : \quad k^i k^j \Gamma^\mu_{ij}(x^0, k^1\tau, k^2\tau, k^3\tau) = 0. \tag{9}$$

(c) Use

$$(\nabla_T E_i)^\nu = T^\mu \partial_\mu (E_i)^\nu + \Gamma^\nu_{\lambda\sigma} T^\lambda (E_i)^\sigma = 0 \quad \text{along } \Gamma_0 \tag{10}$$

(cf. the lecture) and deduce that

$$\forall x^0 : \quad \Gamma^\mu_{0i}(x^0, 0, 0, 0) = 0. \tag{11}$$

Question 2 [*Expansion of the metric to second order*]: Find the second order terms in the expansion (4) in terms of the curvature components $R^\mu_{\nu\lambda\sigma}$.

Hints: Use the fact that the metric is covariantly constant and derive

$$\partial_\alpha \partial_\mu g_{\nu\lambda} = (\partial_\alpha \Gamma^\sigma_{\mu\nu}) g_{\sigma\lambda} + (\partial_\alpha \Gamma^\sigma_{\mu\lambda}) g_{\nu\sigma} \quad \text{along } \Gamma_0. \tag{12}$$

The problem therefore reduces to determining $\partial_\alpha \Gamma^\sigma_{\mu\nu}$ in terms of $R^\mu_{\nu\sigma\lambda}$. This can be done using the following considerations.

(i) Use that

$$R^\mu_{\nu\kappa\lambda} = \partial_\kappa \Gamma^\mu_{\lambda\nu} - \partial_\lambda \Gamma^\mu_{\kappa\nu} \quad \text{along } \Gamma_0. \tag{13}$$

Now define

$$S^\mu_{\kappa\lambda\nu} := \partial_\kappa \Gamma^\mu_{\lambda\nu} + \partial_\lambda \Gamma^\mu_{\nu\kappa} + \partial_\nu \Gamma^\mu_{\kappa\lambda} \tag{14}$$

($S^\mu_{\kappa\lambda\nu}$ is totally symmetric in the lower indices) and derive

$$3\, \partial_\kappa \Gamma^\mu_{\nu\lambda} = R^\mu_{\nu\kappa\lambda} + R^\mu_{\lambda\kappa\nu} + S^\mu_{\kappa\lambda\nu} \quad \text{along } \Gamma_0. \tag{15}$$

Use the hint (b) from question 1 to show that $S^\mu_{ijk} = 0$ along Γ_0.

(ii) Use the hints from question 1 to show that $\partial_0 \Gamma^\mu_{00} = \partial_0 \Gamma^\mu_{0i} = \partial_0 \Gamma^\mu_{ij} = 0$ on Γ_0.

(iii) Use (13) and the results for $\partial_0 \Gamma^\mu_{ij}$ and $\partial_0 \Gamma^\mu_{0i}$ to find $\partial_i \Gamma^\mu_{0j}$ and $\partial_i \Gamma^\mu_{00}$ along Γ_0.