Sheet II

Due: week of October 8

Question 1 [*Receding observers in Minkowski spacetime*]:

Consider the strict timelike future of the origin of Minkowski spacetime given by the set of spacetime events

$$I_0^+ := \left\{ (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 : -(x^0)^2 + \sum_{i=1}^{3} (x^i)^2 < 0 \quad \text{and} \quad x^0 > 0 \right\}. \tag{1}$$

Look at observers in uniform motion, i.e. they move along straight lines through the origin.

(i) At a given event p of an observer, construct the set Σ_p where Σ_p is the set of all nearby events that the observer considers simultaneous with the event p.

(ii) Show that the distribution

$$\Delta := \{\Sigma_p : p \in I_0^+\} \tag{2}$$

is integrable.

Hint: Consider the hyperboloids H_τ^+ given by

$$H_\tau^+ := \left\{ (x^0, x^1, x^2, x^3) \in I_0^+ : (x^0)^2 - \sum_{i=1}^{3} (x^i)^2 = \tau^2 \right\} \tag{3}$$

and show that for a given event $p \in H_\tau^+$ the tangent plane at p to H_τ^+ coincides with Σ_p.

(iii) Consider H_1^+. Assign to each event $p \in H_1^+$ the polar normal coordinates (χ, θ, ϕ), where θ and ϕ are the polar and azimuthal angle of the usual polar coordinates and χ is the distance on the geodesic from the event $(1, 0, 0, 0)$ to the event p. The geodesic being a geodesic of H_1^+. Show that the induced metric on H_1^+ is given by

$$d\sigma^2 = d\chi^2 + \sinh^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2). \tag{4}$$

Show that the Gauss curvature of H_1^+ is equal to -1.

Hints: Let N be the vector given by

$$N^0 = 0, \qquad N^1 = \sin\theta \cos\phi, \qquad N^2 = \sin\theta \sin\phi, \qquad N^3 = \cos\theta \tag{5}$$

and let l be the line given by the direction of N. Consider the plane given by l and the x^0-axis. Rotate the coordinate system such that the line l points in the direction of the x^3-axis. Therefore the plane now coincides with the x^0-x^3-plane. The intersection of H_1^+ with this plane is

$$(x^0)^2 - (x^3)^2 = 1, \qquad x^1 = x^2 = 0. \tag{6}$$

Set

$$x^0 = \cosh\chi, \qquad x^3 = \sinh\chi \tag{7}$$

and check that χ is the distance mentioned in the question.

To compute the Gauss curvature consider a χ-θ-section. Choose this section such that $\theta = \pi/2$. Compute the circumference L of a closed circle in this section. Determine the radius $r := L/2\pi$. The Gauss curvature is

$$-\frac{dk}{d\chi} - k^2, \tag{8}$$

where

$$k := \frac{d}{d\chi}\log r(\chi). \tag{9}$$

(iv) Show that the full spacetime metric is given by

$$ds^2 = -d\tau^2 + \tau^2 d\sigma^2, \tag{10}$$

where $\tau^2 d\sigma^2$ is the metric on H_τ^+. Show that the Gauss curvature of H_τ^+ is equal to $-\tau^{-2}$.

(v) Consider two observers. Consider light emitted by one of the observers and recieved by the other observer. Let χ be given by

$$\tanh\chi = v, \tag{11}$$

where v is the velocity of the observer emitting the light as measured in the rest frame of the observer recieving the light. Show that

$$z = e^\chi - 1, \tag{12}$$

where z is the redshift given by

$$\frac{\omega_e}{\omega_r} - 1, \tag{13}$$

where ω_e and ω_r are the frequencies of the emitted and the recieved light respectively.

Hint: Consider the situation in a reference frame in which the observer recieving the light is stationary, i.e. he moves along the x^0-axis. This can always be achieved by performing an appropriate Lorentz transformation. Then consider the situation in the x^0-x^3-plane ($x^1 = x^2 = 0$). In this plane we have

$$x^0 = \tau\cosh\chi, \qquad x^3 = \tau\sinh\chi \tag{14}$$

and χ coincides with the χ-coordinate of the moving observer.