Sheet I

Due: week of October 1

Question 1 [*Ratio of volumes*]:

As in the first week's lectures we look at the motion of test masses in a gravitational field in the Newtonian theory. We look at a sphere of volume V surrounding a reference particle and being initially at rest relative to the reference particle. We assume that the reference particle (together with the sphere) never enters matter (i.e. always moves through vacuum). Show that the ratio of volumes

$$\alpha(t) := \frac{V(t)}{V(0)} \tag{1}$$

is equal to one up to fourth order terms in t, i.e.

$$\alpha(t) = 1 + O(t^4). \tag{2}$$

Hint: Use the differential equation for $\mu := \log(\alpha)$ from the lecture and derive

$$\frac{d^3\mu}{dt^3}(0) = 0. \tag{3}$$

Question 2 [*Convergence of particles*]:

We assume the same setting as in Question 1. In addition we assume that the Hessian matrix of the gravitational potential does not vanish initially. Recall the definition of the matrix B

$$B := \frac{dA}{dt} A^{-1}, \tag{4}$$

where A is the matrix satisfying

$$Y(t) = A(t)Y(0) \tag{5}$$

and $Y(t)$ is a tangent vector to a curve made up of particles at the position of the reference particle (as in the first week's lectures). Show that there is such a tangent vector $Y(0) \neq 0$ and a time $t^* > 0$ such that

$$Y(t^*) = 0. \tag{6}$$

Interpret this result physically.

This can be done in the following steps:

(i) Show that

$$\frac{d^3 \mathrm{tr}(B)}{dt^3}(0) < 0. \tag{7}$$

Hint: As in question 1, use the differential equation for μ and derive

$$\frac{d^3\mathrm{tr}(B)}{dt^3}(0) = -2|M(0)|^2 < 0, \tag{8}$$

where M denotes the Hessian matrix of the gravitational potential and

$$|M|^2 = \sum_{i,j}(M_{ij})^2. \tag{9}$$

(ii) Taking the trace of the differential equation for B (cf. the first week's lectures), we get

$$\frac{d}{dt}\mathrm{tr}(B) = -\mathrm{tr}(B^2). \tag{10}$$

Show that

$$\frac{d}{dt}\mathrm{tr}(B) \le -\frac{1}{3}(\mathrm{tr}(B))^2. \tag{11}$$

Hints: Prove that B is symmetric by considering the expression for $d\Omega/dt$ together with the initial condition $\Omega(0)$, where $\Omega := B - \tilde{B}$. Here $\tilde{B}$ denotes the transpose of the matrix B. Look at the matrix B in diagonal form and deduce that

$$\mathrm{tr}(B^2) \ge \frac{1}{3}(\mathrm{tr}(B))^2. \tag{12}$$

(iii) Show that there exists a time $t^* > 0$ such that

$$\mathrm{tr}(B) \to -\infty \quad \text{as} \quad t \to t^*. \tag{13}$$

Hint: Define the quantity $u := -1/\mathrm{tr}(B)$ and use (11) to show that

$$u(t) \le u(t_0) - \frac{1}{3}(t - t_0) \tag{14}$$

for $t > t_0 > 0$ and t_0 sufficiently small.

(iv) Show that there exists a tangent vector $Y(0) \neq 0$ such that

$$Y(t^*) = 0. \tag{15}$$