Statistical Physics
Exercise Sheet 4

Exercise 4.1 Local equilibrium state of a gas in a periodic potential

We consider a gas of N particles trapped in a box, $\vec{r} \in V = [0, L]^3$, in the presence of a conservative force $\vec{F}(\vec{r}) = -\nabla\mathcal{V}(\vec{r})$ originating from a periodic potential in the x-direction,

$$\mathcal{V}(\vec{r}) = \mathcal{V}_0 \cos(2\pi k x/L) , \quad k \in \mathbb{N}. \tag{1}$$

For the distribution function in equilibrium, we make the following ansatz; we generalize the free-particle (Maxwell-Boltzmann) distribution function by taking into account a local density $n(\vec{r})$, i.e.,

$$f_0(\vec{r}, \vec{p}) = \frac{n(\vec{r})}{(2\pi m k_\mathrm{B} T)^{3/2}} e^{-\beta p^2/2m} , \quad \beta = \frac{1}{k_\mathrm{B} T}. \tag{2}$$

a) Find the local density $n(\vec{r})$. Discuss the limits $\beta\mathcal{V}_0 \ll 1$ and $\beta\mathcal{V}_0 \gg 1$.

b) Determine the internal energy U and the specific heat C_V. Discuss these expressions in the limits $\beta\mathcal{V}_0 \ll 1$ and $\beta\mathcal{V}_0 \gg 1$.

c) Calculate the entropy $S = S(T, V, N)$.

Hints: The integral representation and the series expansion of the modified Bessel functions of the first kind for $n \in \mathbb{Z}$ are

$$I_n(z) = \frac{1}{\pi} \int_0^\pi d\theta \, e^{z \cos\theta} \cos(n\theta) = \left(\frac{z}{2}\right)^n \sum_{k \geq 0} \frac{(z^2/4)^k}{k!(n+k)!} .$$

The asymptotic behavior for $z \to \infty$ is

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{4n^2 - 1}{8z} + \ldots\right] .$$

Furthermore, the relation $I_0'(z) = I_1(z)$ holds.

Exercise 4.2 Distribution function for relativistic massive particles

Consider a system of relativistic massive particles with vanishing drift velocity $\vec{v}_\mathrm{drift} = 0$. The energy-momentum relation of the particles writes

$$E(\vec{p}) = \sqrt{p^2 c^2 + m^2 c^4} , \tag{3}$$

where m is the particle mass and c the speed of light.

a) Find the equilibrium distribution function $f_0(\vec{p})$ and show that in the limit $k_\mathrm{B} T \ll mc^2$, the classical Maxwell-Boltzmann distribution function is recovered. Calculate the internal energy U and determine the first relativistic corrections to this expression as well as to the specific heat C_V.

b) Calculate the pressure P for this system and convince yourself that in the limit $k_{\mathrm{B}}T/mc^2 \ll 1$, it is consistent with the pressure of a dilute ideal gas as derived in the lecture notes.

c) Take into account a finite drift velocity $\vec{v}_{\mathrm{drift}}$. How does the distribution function change? Does the temperature depend on the drift velocity? Calculate the average momentum $\langle \vec{p} \rangle$ for this system.

Hints: The integral representation of the modified Bessel functions of the second kind is

$$K_n(z) = \int_0^\infty dy\, e^{-z\cosh y} \cosh(ny) \,,$$

Its asymptotic behavior for $z \to \infty$ is

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \Big[1 + \frac{4n^2-1}{8z} + \frac{(4n^2-1)(4n^2-9)}{2!(8z)^2} + \frac{(4n^2-1)(4n^2-9)(4n^2-25)}{3!(8z)^3} + \ldots \Big] \,.$$

Office Hours: Monday, October 15th, 8–10 AM (Michael Walter, HIT K 31.5).