Dissipation in Quantum Systems Exercise 6

HS 12
Dr. R. Chitra

Exercise 6.1 Fluctuations and decoherence

Consider a two-level system, e.g. a spin-1/2 particle, coupled to a bath. We want to study the effect of the bath on the dynamics of the spin-1/2 which we call the *central spin*. Often, the bath consists of spins (spins of the atom cores) and/or of bosons (phonons in the solid).
For simplicity, we assume that the spin/boson baths do not interact. In order to understand the physics we want to study the decoherence of the central spin.
The Hamiltonian of the central spin coupled to a bath of N spin-1/2 particles is given by

$$H = \sum_{i=1}^{N} h_i \sigma_i^z - \sigma_c^x \sum_{i=1}^{N} \lambda_i \sigma_i^x \tag{1}$$

where $\sigma^{x,z}$ are the Pauli matricies (σ_i corresponds to the spin i of the bath and σ_c to the central spin), h_i is a field acting on the spin i and λ_i is the coupling parameter between the central spin and the spins of the bath. We work in the basis $\{|\leftarrow\rangle, |\rightarrow\rangle\}$, i.e. the eigenstates of σ_c^x with

$$\sigma_c^x|\leftarrow\rangle = -|\leftarrow\rangle \qquad \sigma_c^x|\rightarrow\rangle = |\rightarrow\rangle. \tag{2}$$

a) We prepare the central spin at $t = 0$ in the pure initial state

$$|\psi\rangle = \alpha|\leftarrow\rangle + \beta|\rightarrow\rangle. \tag{3}$$

Calculate the density matrix $\rho(t = 0)$ of the central spin!

b) Assume that the bath is in thermal equilibrium with the temperature T. What is the density matrix ρ_B of the bath?

c) The density matrix of the whole system is given by

$$\Omega = \rho(0) \otimes \rho_B. \tag{4}$$

The central spin for arbitrary time t is described by the reduced density matrix

$$\rho(t) = \mathrm{Tr}_B\left(e^{-iHt}\Omega e^{iHt}\right) \tag{5}$$

where the index B denotes the trace over all degrees of freedom of the bath. Show that the reduced density matrix is given by

$$\rho(t) = |\alpha|^2|\leftarrow\rangle\langle\leftarrow| + |\beta|^2|\rightarrow\rangle\langle\rightarrow| + M(t)\alpha^*\beta|\leftarrow\rangle\langle\rightarrow| + M^*(t)\alpha\beta^*|\rightarrow\rangle\langle\leftarrow|. \tag{6}$$

Find an expression for $M(t)$, the decoherence factor!

d) Calculate $M(t)$ in the case $h_i \equiv 0$. Use the substitution $\lambda_i = \lambda/\sqrt{N}$ with the total number of spins N in the bath and show that $M(t)$ tends to a Gaussian.

e) For $h_i \neq 0$, show that

$$\ln(M(t)) = \sum_{k=1}^{N} \ln\left[1 - \frac{2\lambda_k^2}{\lambda_k^2 + h_k^2} \cdot \sin^2\left(t\sqrt{\lambda_k^2 + h_k^2}\right)\right] \tag{7}$$

and conclude!