Dissipation in Quantum Systems Exercise 5

HS 12
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Exercise 5.1 Harmonic Oscillator II

We consider a lattice consisting of dissipative harmonic oscillators which are coupled to each other. The lattice constant a is the distance between neighbouring lattice sites and D is the dimensionality of the lattice. Furthermore, we have the mass density μ, the elasticity constant c and a characteristic time τ due to damping of each individual oscillator. The displacement of the oscillator from its equilibrium position $\vec{r}_i$ is denoted as u_i.
The equation of motion for this system is then given by

$$\frac{d^2u_i}{dt^2} + \frac{1}{\tau}\frac{du_i}{dt} + \frac{c}{\mu a^2}\sum_{j(i)}(u_i - u_j) = 0 \tag{1}$$

where $j(i)$ denotes all sites j which are nearest neighbours of the site i.

a) Try the following ansatz

$$u_i(t) = Ae^{-\gamma t}e^{i\vec{k}\vec{r}_i - i\Omega_{\vec{k}}t} \tag{2}$$

in order to solve the coupled differential equation (1). For which $\vec{k}$ exists an oscillating solution, i.e. $\Omega_{\vec{k}}$ is real? What is the value of γ?

Show that $\Omega_{\vec{k}}$ in the continuum limit $a \to 0$ is given by

$$\Omega_{\vec{k}} = \pm\sqrt{\omega_{\vec{k}}^2 - \gamma^2} \tag{3}$$

where $\gamma = 1/2\tau$ and $\omega_{\vec{k}} = v|\vec{k}|$ with $v = \sqrt{c/\mu}$.

What happens to the solutions $u_i(t)$ which correspond to forbidden wave vectors $\vec{k}$?

b) Compute the equal time correlation function $\langle [u_i(t) - u_j(t)]^2 \rangle$ for the classical limit ($\omega_{\vec{k}} \gg \gamma$) at temperature T!

Hint: Write $u_i(t)$ in its Fourier components $u_{\vec{k}}(t)$, i.e.

$$u_i(t) = \frac{1}{\sqrt{N}}\sum_{\vec{k}} u_{\vec{k}}(t)e^{i\vec{k}\vec{r}_i}, \tag{4}$$

and use/motivate the following relation using (a) and Exercise 4,

$$\langle u_{\vec{k}}(t)u^*_{\vec{k}'}(t) \rangle = \delta_{\vec{k},\vec{k}'} \cdot C_{xx}(t-t) = \delta_{\vec{k},\vec{k}'} \cdot \frac{k_B T}{a^D \mu \Omega^2_{\vec{k}}}. \tag{5}$$

c) Motivate that the equal time correlation function in the quantum limit at $T = 0$ is roughly given by

$$\langle [u_i(t) - u_j(t)]^2 \rangle \approx \frac{\hbar}{Na^D}\sum_{\vec{k}}\left(\frac{\theta(\omega_{\vec{k}} - \gamma)}{2\mu\omega_{\vec{k}}} + \frac{\theta(\gamma - \omega_{\vec{k}})}{\pi\mu\gamma}\log\left(\frac{2\gamma}{\omega_{\vec{k}}}\right)\right) \cdot 2\left[1 - \cos\left(\vec{k}(\vec{r}_i - \vec{r}_j)\right)\right]. \tag{6}$$

Hint: Treat the cases $\gamma > \omega_{\vec{k}}$ (strong damping) and $\omega_{\vec{k}} > \gamma$ (weak damping) independently. The case of a single strongly damped quantum harmonic oscillator was already subject of Exercise 4 where we obtained

$$C_{xx}(t=0) = \frac{\hbar}{\pi m \gamma} \log\left(\frac{2\gamma}{\omega_0}\right) \tag{7}$$

where ω_0 was the eigenfrequency. In the weak damping regime we neglect γ completely and use the equipartition principle for a harmonic oscillator in its quantum version

$$\left\langle \frac{1}{2} m \omega_0^2 x^2 \right\rangle = \frac{\hbar \omega_0}{4} \tag{8}$$

in order to estimate C_{xx}.

d) In the case $D = 1$, dissipation introduces a new characteristic length scale ξ. Determine its value and study $\langle [u_i(t) - u_j(t)]^2 \rangle$ for $a \ll |\vec{r}_i - \vec{r}_j| \ll \xi$! What happens in the limit $|\vec{r}_i - \vec{r}_j| \to \infty$? Discuss the results!

e)** Is dissipation important for spatial dimensionality $D \geq 2$? In which physical systems this mechanism might play an important role? Argue!