

## 0 Overview

String theory is an attempt to quantise gravity and unite it with the other fundamental forces of nature. It combines many interesting topics of (quantum) field theory in two and higher dimensions. This course gives an introduction to the basics of string theory.

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9. String Backgrounds
10. Superstrings
11. AdS/CFT Correspondence

### 0.2 References

There are many text books and lecture notes on string theory. Here is a selection of well-known ones:

- classic: M. Green, J.H. Schwarz and E. Witten, “Superstring Theory” (2 volumes), Cambridge University Press (1988)
- alternative: D. Lüst, S. Theisen, “Lectures on String Theory”, Springer (1989).
- standard: J. Polchinski, “String Theory” (2 volumes), Cambridge University Press (1998)
- basic: B. Zwiebach, “A First Course in String Theory”, Cambridge University Press (2004/2009)
- recent: K. Becker, M. Becker, J.H. Schwatz, “String Theory and M-Theory: A Modern Introduction”, Cambridge University Press (2007)
- online: D. Tong, “String Theory”, lecture notes, <http://arxiv.org/abs/0908.0333>
- ...

# 1 Introduction

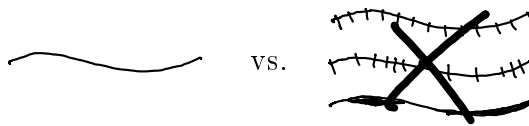
## 1.1 Definition

String theory describes the mechanics of one-dimensional extended objects in an ambient space. Some features:

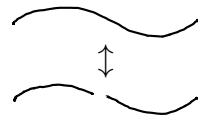
- Strings have tension:



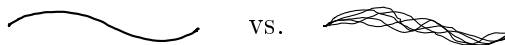
- Strings have no inner structure:



- Several pieces of string can interact:



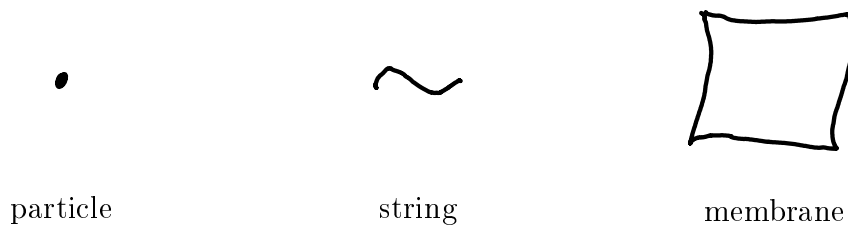
- Strings can be classical or quantum:



## 1.2 Motivation

Why study strings?

**Extended Objects.** We know a lot about the mechanics of point particles. It is natural to study strings next. Or even higher-dimensional extended objects like membranes...



particle

string

membrane

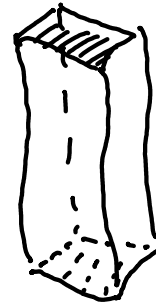
These are objects are snapshots at fixed time  $t$ . Introduce the worldvolume as the volume of spacetime occupied by the object:



worldline



worldsheet



worldvolume

The worldsheet of a string is two-dimensional. In fact, there is a great similarity between strings and static soap films.

**Quantum Gravity.** String theory offers a solution to the problem of quantum gravity (QG). (really?) Sketch without reference to quantum field theory (QFT).

Classical gravity theories:

- Newtonian Gravity (non-relativistic)
- General Relativity (GR, relativistic, geometry of spacetime)

Nature is quantum mechanical, gravity must also be quantum. Need QG for: early universe, black hole radiation. Field quantisation introduces quanta (particles):

- electromagnetism: photon
- strong nuclear forces: gluons
- gravity: graviton

Particles interact through vertices (Feynman rules). Relatively simple rules for standard model

$$S = \text{[diagram of a line with a dot]} + \text{[diagram of a Y-vertex]} + \text{[diagram of an X-vertex]} .$$

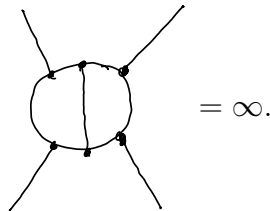
Einstein gravity has infinitely many vertices

$$S = \text{[diagram of a line with a dot]} + \text{[diagram of a Y-vertex]} + \text{[diagram of an X-vertex]} + \text{[diagram of a 5-vertex]} + \text{[diagram of a 6-vertex]} + \dots$$

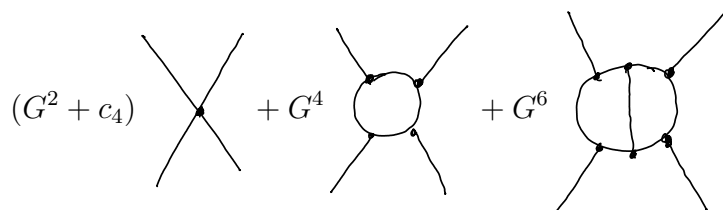
In fact, can introduce additional couplings  $c_k$ :

$$G \text{ [diagram of a Y-vertex]} + (G^2 + c_4) \text{ [diagram of an X-vertex]} + (G^3 + c_5) \text{ [diagram of a 5-vertex]} + \dots$$

Classically we do not need the  $c_k$ , but in QFT we do. Feynman loops generate divergences, e.g.



Need to sum up all competing processes:



Divergence can be absorbed into  $c_4 = -G^6\infty + c_{4,\text{ren}}$ .

All well, but no good way to set renormalised  $c_{4,\text{ren}}$  to zero. Unfortunately, divergences require infinitely many  $c_k$ . Infinitely many adjustable parameters, not predictive! Only good prediction at low energies, densities: GR.

What does string theory have to do with it?

Quantum string theory turns out to contain gravitons. Moreover, generates no divergences; finite! String theory has just a few coupling constants.

All well!?

Almost, there may be many more couplings elsewhere.

**Unification.** String theory provides a unified description for all kinds of fundamental forces of nature. (the correct one?)

Electromagnetic and weak forces combine into electroweak forces at sufficiently high energies  $10^2\text{GeV}$ . Also with strong forces (Grand Unified Theory, GUT)?

Hints:

- Charges of fermions appear to suggest larger gauge group:

$$\text{SU}(3) \times \text{SU}(2) \times \text{U}(1) \longleftarrow \text{SU}(5), \text{SO}(10)?$$

- Estimated GUT scale  $10^{15}\text{GeV}$  near Planck scale  $10^{18}\text{GeV}$ . Suggests unification of all forces.
- Wouldn't it be nice?

String theory describes gauge theories as well as gravity. In particular, group sequence  $\text{SU}(5), \text{SO}(10), \dots$  appears.

Does it describe nature? So far no convincing derivation. Best option: Standard Model (SM) among many(!) "natures".

**String/Gauge Duality.** Intricate relations between string and gauge theories (used in SM).

In some cases gauge theory *is* string theory.

String theory is part of gauge theory, not just QG.

**Treasure Chest.** String theory yields many interesting, novel, exceptional structures, results, insights in physics and mathematics. Just to name a few: supersymmetry, higher dimensions,  $p$ -branes, dualities, topological insights.

**Many Unsolved Problems.** (despite 40 years of research)

- How to match with nature?
- How to find direct/indirect evidence? (Susy?)
- What is String Theory?
- How to quantise gravity (otherwise)?

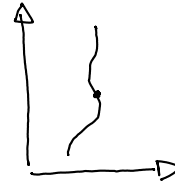
## 2 Relativistic Point Particle

Let us start slowly with something else: a relativistic particle. Here we will encounter several issues of string theory, but in a more familiar setting. There are many formulations, we will discuss several.

### 2.1 Non-Relativistic Actions

First: a free non-relativistic point particle  $\vec{x}(t)$ . Action with resulting equations of motion (e.o.m.):

$$S[\vec{x}] = \int dt \frac{1}{2} m \dot{\vec{x}}(t)^2, \quad \ddot{\vec{x}}(t) = 0.$$



Momentum and energy from Hamiltonian formulation:

$$\vec{p}(t) = \frac{\partial S[\vec{x}]}{\partial \dot{\vec{x}}(t)} = m \dot{\vec{x}}(t), \quad E(t) = H(t) = \frac{\vec{p}(t)^2}{2m}.$$

Note: functional variation removes the integral.

Promote the above to a relativistic particle:

$$S = - \int dt mc \sqrt{c^2 - \dot{\vec{x}}^2} \approx \int dt \left( -mc^2 + \frac{1}{2} m \dot{\vec{x}}^2 + \frac{1}{8} mc^{-2} \dot{\vec{x}}^4 \right).$$

Derive e.o.m.:

$$(c^2 - \dot{\vec{x}}^2) \ddot{\vec{x}} + (\dot{\vec{x}} \cdot \ddot{\vec{x}}) \dot{\vec{x}} = 0.$$

They imply collinearity  $\ddot{\vec{x}} = \alpha \dot{\vec{x}}$ . Substitute to get  $\alpha c^2 \dot{\vec{x}} = 0$  hence  $\ddot{\vec{x}} = 0$ .

Momentum and energy read

$$\vec{p} = \frac{mc \dot{\vec{x}}}{\sqrt{c^2 - \dot{\vec{x}}^2}}, \quad E = c \sqrt{m^2 c^2 + p^2}.$$

Fine, but is not manifestly relativistic: Non-relativistic formulation of a relativistic particle. Want a manifestly relativistic formulation using 4-vectors  $X^\mu = (ct, \vec{x})$  and  $P_\mu = (E/c, \vec{p})$ . Let us set  $c = 1$  for convenience from now on.

- Momentum  $P_\mu$  is already a good 4-vector:

$$P^2 = -E^2 + \vec{p}^2 = -m^2.$$

Mass shell condition  $P^2 = -m^2$  manifestly relativistic. But origin/role of  $\vec{p}$  and  $E$  is quite distinct.

- Position  $X^m(t) = (t, \vec{x}(t))$  and action make explicit reference to time  $t$  (in a particular Lorentz frame)

$$S = - \int dt m \sqrt{- \left( \frac{dX(t)}{dt} \right)^2}.$$

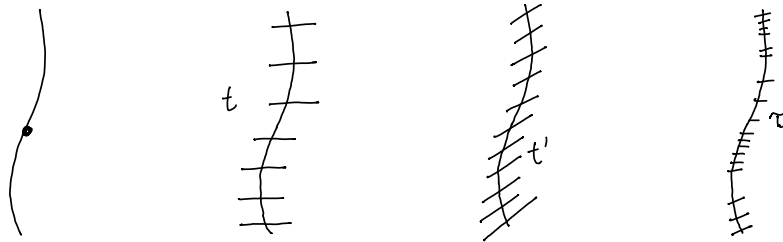
- Note that Hamiltonian framework distinguishes between space and time: Explicit reference to time derivatives.

## 2.2 Worldline Action

Notice: above action measures Lorentz-invariant proper time  $s$  of the particle's path  $X^\mu(t)$  in spacetime (worldline)

$$S = -m \int ds, \quad \text{where } ds^2 = -dX^2.$$

Proper time depends only on the location of the worldline, but not on a particular Lorentz frame (definition of  $t$ ) or parametrisation of the worldline (through  $t$ ).



Let us assume an arbitrary parametrisation  $X^\mu(\tau)$  of the worldline through some curve parameter  $\tau$ . The proper time action reads (now dot denotes  $d/d\tau$ )

$$S = - \int d\tau m \sqrt{- \left( \frac{dX(\tau)}{d\tau} \right)^2} = - \int d\tau m \sqrt{-\dot{X}^2}.$$

Nice manifestly relativistic formulation. Notice: 4 undetermined functions  $X^\mu(\tau)$  instead of 3 undetermined functions  $\vec{x}(t)$ ; new function  $t(\tau)$ .

Use this as starting point, derive equations of motion

$$\dot{X}^2 \ddot{X}^\mu = (\dot{X} \cdot \ddot{X}) \dot{X}^\mu.$$

Implies collinearity  $\ddot{X}^\mu = c \dot{X}^\mu$  for all  $\tau$  with variable  $c(\tau)$ . Meaning: worldline straight. ✓

Next derive momenta as derivatives of  $S$  w.r.t.  $\dot{X}^\mu$

$$P_\mu = \frac{m \dot{X}_\mu}{\sqrt{-\dot{X}^2}}.$$

Obey mass shell condition  $P^2 = -m^2!$  ✓

Only three independent  $P_\mu$  but four independent  $X^\mu!$

Moreover naive Hamiltonian is strictly zero  $H = 0!$

Signals presence of constraint and gauge invariance:

- Reparametrising  $\tau' = f(\tau)$  has no effect on physics.
- Redundancy of description: worldline coordinate  $\tau$ .
- One linear dependency among the e.o.m. for  $X^\mu$ .
- Gauge invariance effectively removes one  $X^\mu$ , e.g. time  $t(\tau)$ .
- Situation inconvenient for Hamiltonian framework/QM.
- Better to fix a gauge, many choices, pick a convenient one.

Obtained a fully relativistic formulation, but packaged with complication of gauge invariance. In fact, gauge invariance often considered a virtue: Symmetry!

Above worldline action has two further drawbacks:

- Is non-polynomial; inconvenient for quantisation.
- Does not work for massless particles  $m = 0$ .

## 2.3 Polynomial Action

There is an equivalent action with an auxiliary variable  $e(\tau)$

$$S = \int d\tau \left( \frac{1}{2} e^{-1} \dot{X}^2 - \frac{1}{2} e m^2 \right).$$

The resulting e.o.m. read

$$m^2 e^2 + \dot{X}^2 = 0, \quad e \ddot{X}^\mu - \dot{e} \dot{X}^\mu = 0.$$

In combination they yield the same old equation for  $X^\mu$ . The momentum conjugate to  $X^\mu$  reads  $P_\mu = e^{-1} \dot{X}_\mu$ , hence the equation of motion for  $e$  reduces to  $P^2 = -m^2$ . Momentum conjugate to  $e$  vanishes signalling a constraint.

Massless case  $m = 0$  works at every step of above derivation, yields constant  $P^\mu = e^{-1} \dot{X}^\mu$  as well as  $P^2 = 0$ . Notice:  $e$  not fixed by e.o.m.; commonly gauge freedom remains for massless case.

Field  $e$  has a nice geometrical interpretation: Einbein specifying a metric  $g_{\tau\tau} = -e^2$  on the worldline. All terms in the action are in the right combination; remain invariant under changing worldline coordinates ( $e$  transforms according to  $e' = e d\tau'/d\tau$ ).

Here einbein  $e$  is a dynamical variable. Curiously, e.o.m. picks out metric induced by ambient space. When substituting solution for  $e$  recover action of Sec. 2.2.

## 2.4 Various Gauges

We have freedom to fix one of the coordinates  $X^\mu(\tau)$  at will. Some more or less useful choices:



- **Temporal Gauge.**  $t(\tau) = \tau$  or  $t(\tau) = \alpha\tau$ .  
Reduces to non-relativistic treatment of Sec. 2.1.
- **Spatial Gauge.**  $z(\tau) = \alpha\tau$ .  
Works locally except at turning points of  $z(\tau)$ .
- **Light Cone Gauge.**  $x^+(\tau) := t(\tau) + z(\tau) = \alpha\tau$ .  
Useful in some cases; prominent in string theory.
- **Proper Time Gauge.**  $ds = d\tau$ .  
Fixes  $t(\tau)$  through integral

$$t(\tau) = \int^{\tau} d\tau' \sqrt{1 + \dot{x}(\tau')^2}.$$

Action becomes trivial  $S = - \int d\tau$ ; deal with constraint.

- **Constant Einbein.**  $\dot{e} = 0$ .  
In polynomial formulation, gauge fixing may involve  $e$ . Customary gauge choice is constant  $e$ . E.o.m. reduces to

$$\ddot{X} = 0.$$

We replace dynamical variable  $e$  by a constant; we must remember its equation of motion

$$\dot{X} + m^2 e^2 = 0.$$

In gauge fixed formulation it becomes constraint.

## 2.5 Quantisation

Quantisation can be done in several different ways. Depends on the choice of classical formulation. Let us pick polynomial action discussed in Sec. 2.3. For the Hamiltonian formulation it is best to also fix a gauge; we will choose the einbein  $e$  to be constant. Momenta  $P$  associated to  $X$  and resulting Hamiltonian read:

$$P = e^{-1} \dot{X}, \quad H = \frac{1}{2} e (P^2 + m^2).$$

Conventionally, a state  $|\Psi\rangle$  is given by a wave function of position variables and time

$$|\Psi\rangle = \int d^4 X \Psi(X, \tau) |X\rangle.$$

Slightly more convenient to immediately Fourier transform to momentum space  $|X\rangle \simeq \int d^4 P e^{iP \cdot X} |P\rangle$

$$|\Psi\rangle = \int d^4 P \Psi(P, \tau) |P\rangle, \quad \Psi(P, \tau) = \int d^4 X e^{iP \cdot X} \Psi(X, \tau).$$

Schrödinger equation reads

$$i\dot{\Psi} = H\Psi = \frac{i}{2} e (P^2 + m^2) \Psi.$$

Obviously, solved by

$$\Psi(P, \tau) = \exp\left(-\frac{i}{2} e (P^2 + m^2) \tau\right) \Phi(P).$$

Need to remember that system is constrained; wave function must vanish whenever constraint not satisfied:

$$(P^2 + m^2)\Psi(P, \tau) = 0.$$

In effect, physical states  $\Psi(P, \tau) = \Phi(P)$  are independent of  $\tau$ . Makes perfect sense: worldline coordinate  $\tau$  unphysical. Schrödinger equation governing  $\tau$ -evolution replaced by constraint  $P^2 + m^2 = 0$  (governing  $t$ -evolution).

Fourier transform wave function back to position space  $\Phi(X)$ ; constraint becomes the Klein–Gordon equation for spin-0 field

$$(-\partial^2 + m^2)\Phi(X) = 0.$$

## 2.6 Interactions

Obviously, free particle is easy; eventually would like to include interactions. Let us sketch how to add interactions with external potentials and with other particles:

**Electrical and Gravitational Fields.** Coupling to electrical and gravitational fields takes a very geometric form

$$S = \int d\tau \left( \frac{1}{2} e^{-1} g_{\mu\nu}(X) \dot{X}^\mu \dot{X}^\nu - \frac{1}{2} e m^2 + A_\mu(X) \dot{X}^\mu \right).$$

$A_\mu$  potential for the electromagnetic field  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ . Likewise  $g_{\mu\nu}$  is the gravitational potential; takes the form of the metric of a curved spacetime.

These are fixed external fields: Unaffected by presence of particle, but influence its motion. Note: Fields are evaluated at dynamical position  $X^\mu(\tau)$ .

In quantum mechanics, one usually assumes weak interactions. Formally allows to work with free quantum fields; interactions are introduced in perturbative fashion. When free particle enters potential field, it scatters off of it. Dominant contribution from single scattering; multiple interactions suppressed. Only in rare instances, potentials can be handled exactly.

**Interactions Among Particles.** Local interactions: Several particles meet at some spacetime point and split up, potentially into a different number of particles. In worldline formulation achieved by introducing vertices where several particle worldlines meet:



This is not standard treatment of interaction. Typically interaction of  $n$  fields: term  $\Phi^n$  in QFT action. Our method is not very convenient, but it works as well.

Nevertheless instructive, mimics Feynman rules; It is the standard procedure for string theory.

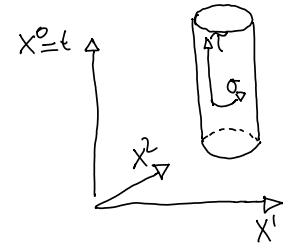
## 2.7 Conclusions

- Seen many equivalent formulations of same physical system.
- Had to deal with gauge invariance and constraints.
- Different number of degrees of freedom (d.o.f.), but number of solutions (modulo gauge) always the same.
- Quantised the free relativistic particle.
- Discussed interactions.
- Not always most convenient path chosen; but treatment of string will be analogous.

### 3 Classical Bosonic String

Mechanics of 1D extended object without inner structure.

- Worldsheet coordinates:  $\xi^\alpha = (\tau, \sigma)$ , time  $\tau$ , space  $\sigma$ .
- Embedding coordinates  $X^\mu(\xi)$ .
- $D$ -dimensional embedding Minkowski space, metric  $\eta_{\mu\nu}$ .



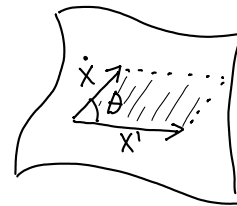
#### 3.1 Nambu–Goto Action

Generalise insights from relativistic point particle:

- worldline  $\rightarrow$  worldsheet.
- action = proper time  $\simeq$  “length”  $\rightarrow$  “area”.

**Area and Action.** Wick rotation  $t = iw$ . Area of 2D euclidean surface:

$$\begin{aligned} dA &= d\tau d\sigma |X'| |\dot{X}| |\sin \theta| \\ &= d\tau d\sigma \sqrt{X'^2 \dot{X}^2 \sin^2 \theta} \\ &= d\tau d\sigma \sqrt{X'^2 \dot{X}^2 - (X' \cdot \dot{X})^2} \\ &= d^2\xi \sqrt{\det g}, \end{aligned}$$



induced worldsheet metric  $g_{\alpha\beta} = \eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu$  (pull back).

Wick rotate back; string action (Nambu–Goto)

$$S = -\frac{1}{2\pi\kappa^2} A = -\frac{1}{2\pi\kappa^2} \int d^2\xi \sqrt{-\det g}.$$

Symmetries of action:

- **Lorentz:** scalar products of Lorentz vectors  $X$ .
- **Poincaré:**  $X$  only through  $\partial X$ .
- **Worldsheet Diffeomorphisms:** density  $d^2\xi \sqrt{-\det g}$  invariant under reparametrisations  $\xi \mapsto \xi'(\xi)$ .

**Tension.** What is parameter  $\kappa$ ?

Fundamental string length scale  $\rightarrow$  quantum string.



Consider potential  $U$ : Time slice of action/area. Slice of length  $L$ :  $U \sim L/\kappa^2$ .  
Constant force  $U' = 1/2\pi\kappa^2 = T$  is string tension. String not a spring or rubber band!

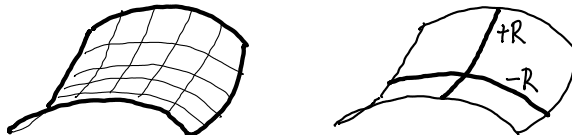
**Equations of Motion.** Vary action.

Use variation of determinant  $\delta \det g = \det g g^{\alpha\beta} \delta g_{\alpha\beta}$ .

$$\partial_\alpha (\sqrt{-\det g} g^{\alpha\beta} \partial_\beta X^\mu) = 0.$$

Highly non-linear equations ( $g$  contains  $X$ )! How to solve? How to deal with?

Geometrically: stationary action; minimal area surface. Static soap films! Mean curvature zero. Saddle point everywhere, equal/opposite sectional curvatures.



### 3.2 Polyakov Action

Complication from non-linear e.o.m.. As for point particle, there is a polynomial action with additional dynamical worldsheet metric  $g_{\alpha\beta}$ :

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi \sqrt{-\det g} \frac{1}{2} g^{\alpha\beta} (\partial_\alpha X) \cdot (\partial_\beta X).$$

E.o.m. for  $X$  as above; e.o.m. for worldsheet metric  $g$ :

$$(\partial_\alpha X) \cdot (\partial_\beta X) = \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} (\partial_\gamma X) \cdot (\partial_\delta X).$$

Solution fixes induced metric (up to local scale  $f$ )

$$g_{\alpha\beta} = f(\xi) (\partial_\alpha X) \cdot (\partial_\beta X).$$

Arbitrary scale  $f(\xi)$  cancels in action and all e.o.m.. New redundancy: **Weyl invariance**  $g_{\alpha\beta}(\xi) \mapsto f(\xi) g_{\alpha\beta}(\xi)$ .

### 3.3 Conformal Gauge

E.o.m. for  $X$  linear, coupling to  $g$  makes non-linear.

Use gauge freedom: demand conformally flat metric.

$$g_{\alpha\beta}(\xi) = f(\xi) \eta_{\alpha\beta}.$$

Amounts to two equations  $g_{\tau\sigma}(\xi) = 0$ ,  $g_{\tau\tau}(\xi) = -g_{\sigma\sigma}(\xi)$ .

- Fixes almost all diffeomorphisms.
- Conformal transformations remain. Diffeomorphisms preserving metric up to scale.
- May further set  $f = 1$  by means of fixing Weyl.

Action describes  $D$  free massless scalar particles in 2D

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi \frac{1}{2} \eta^{\alpha\beta} (\partial_\alpha X) \cdot (\partial_\beta X).$$

E.o.m. simply  $\partial^2 X^\mu = 0$  (harmonic wave equation)

$$\ddot{X} = X''.$$

Do not forget e.o.m. for worldsheet metric

$$T_{\alpha\beta} := (\partial_\alpha X) \cdot (\partial_\beta X) - \frac{1}{2} \eta_{\alpha\beta} \eta^{\gamma\delta} (\partial_\gamma X) \cdot (\partial_\delta X) = 0.$$

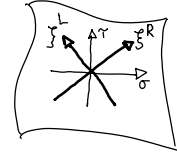
$T_{\alpha\beta}$  is energy-momentum tensor for  $D$  scalars. Trace absent  $T^\alpha_\alpha = 0$  by construction (Weyl/conformal). Two remaining e.o.m. become **Virasoro constraints**

$$\dot{X} \cdot X' = 0, \quad \dot{X} \cdot \dot{X} + X' \cdot X' = 0.$$

Forbids longitudinal waves along the worldsheet, no structure!

Conservation of  $T_{\alpha\beta}$ : impose constraints only on time slice.

### 3.4 Solution on the Light Cone



Solve harmonic wave equation. Light cone coordinates  $\xi^{L/R}$  useful:

$$\xi^{L/R} = \tau \mp \sigma, \quad \partial_{L/R} = \frac{1}{2}(\partial_\tau \mp \partial_\sigma),$$

new worldsheet metric

$$d^2s = -d\tau^2 + d\sigma^2 = -d\xi^L d\xi^R.$$

Now e.o.m. and Virasoro constraints read

$$\partial_L \partial_R X^\mu = 0, \quad (\partial_{L/R} X)^2 = 0.$$

First equation solved by simple separation of variables

$$X^\mu(\xi^L, \xi^R) = X_L^\mu(\xi^L) + X_R^\mu(\xi^R).$$

$D$  left-movers  $X_L$  plus  $D$  right-movers  $X_R$ . Virasoro constraints  $(\partial X_{R,L})^2 = 0$  remove 1 left/right-mover. Two reparametrisations left:

- **conformal transformations**

$$\xi^R \mapsto \xi'^R(\xi^R), \quad \xi^L \mapsto \xi'^L(\xi^L).$$

2D case special: infinitely many transformations. removes another 1 left/right-mover.

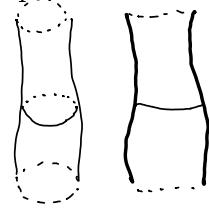
- **constant shift** between  $X_L^\mu$  and  $X_R^\mu$ .

$(D - 2)$  left/right-movers remain (transverse).

### 3.5 Closed String Modes

So far worldsheet infinitely extended in space and time; want finite spatial extent.

- **Closed String:** circular topology.  
Identify  $\sigma \equiv \sigma + 2\pi$  (other choices possible).
- **Open String:** interval topology.  
Boundary conditions at  $\sigma = 0, \pi$  (later).



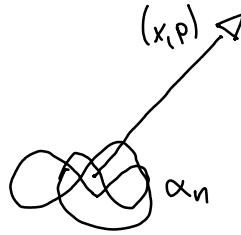
Periodic function  $X \rightarrow$  Fourier decomposition

$$X_{L/R}^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\kappa^2 p^\mu \xi^{L/R} + \sum_{n \neq 0} \frac{i\kappa}{\sqrt{2n}} \alpha_n^{L/R, \mu} \exp(-in\xi^{L/R}).$$

- Coefficients  $i\kappa/\sqrt{2n}$  chosen for later convenience.
- Linear dependence in  $\xi^{L/R}$  okay:  $X^\mu = x^\mu + \kappa^2 p^\mu \tau + \dots$
- Reality of  $X$ : complex conjugate  $\alpha_{-n} = (\alpha_n)^*$ .

Two kinds of parameters for solutions

- Centre of mass motion  $x, p$  (conjugate variables  $\rightarrow \kappa^2$ ).
- String modes  $\alpha_n^{L/R, \mu}$  (left/right movers).



Consider Virasoro constraints, substitute modes:

$$(\partial_{L,R} X_{L/R}^\mu)^2 = \kappa^2 \sum_n L_n^{L/R} \exp(-in\xi^{L/R}) \stackrel{!}{=} 0$$

with Virasoro modes (drop L/R index)

$$L_n := \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m \stackrel{!}{=} 0.$$

Note that  $\alpha_0^L = \alpha_0^R = \kappa p / \sqrt{2}$ .  $L_0 = 0$  constraint fixes string mass

$$p^2 = -M^2, \quad M^2 = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m$$

All Virasoro constraints  $L_n = 0$  conserved by e.o.m.

$$\dot{L}_n = inL_n.$$

Need to impose on initial data only.

String mass depends on mode amplitudes.

- No modes excited: massless point particle.
- Few/small excitations: light tiny particle.
- Many/large excitations: big massive object.



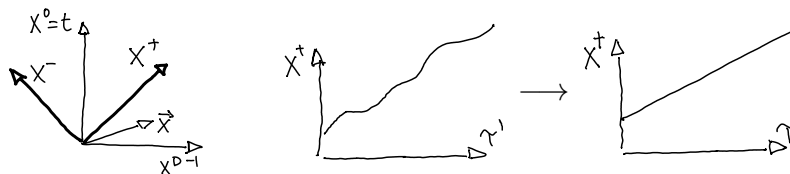
Note: time-like modes  $\alpha^0$  contribute negative  $M^2$ . Tachyons excluded by Virasoro constraints.

Virasoro constraints non-linear, complicated.

### 3.6 Light Cone Gauge

Use conformal symmetry to solve Virasoro constraints.

**Gauge Fixing.** Introduce as well light cone coordinates for spacetime  $X^\pm = X^0 \pm X^{D-1}$ .  $\vec{X}$  denotes components  $1 \dots (D-2)$ . Gauge fix!



$$X_{L/R}^+ = x_{L/R}^+ + \frac{1}{2}\kappa^2 p_{L/R}^+ \xi^{L/R}.$$

Virasoro constraint  $(\partial \vec{X}_{L/R})^2 - \kappa^2 p_{L/R}^+ \partial X_{L/R}^- = 0$  solved by

$$X_{L/R}^-(\xi) = \frac{1}{\kappa^2 p_{L/R}^+} \int^\xi d\xi' (\partial \vec{X}_{L/R}(\xi'))^2.$$

Solution:  $2(D-2)$  arbitrary functions  $\vec{X}_{L/R}(\xi^{L/R})$ .

**Periodicity.** All functions  $\vec{X}(\xi)$  periodic! Furthermore: periodicity of  $X^+$  and  $X^-$  requires

$$p_L^+ = p_R^+ = p^+, \quad \int_0^{2\pi} d\xi ((\partial \vec{X}_R)^2 - (\partial \vec{X}_L)^2) = 0.$$

Residual gauge freedom: Constant shift  $\Delta X_R^\mu(\xi) = -\Delta X_L^\mu(\xi)$ . Corresponds to above residual constraints.



**String Modes.** Impose gauge fixing on modes ( $n \neq 0$ )

$$\alpha_n^+ = 0, \quad \alpha_n^- = \frac{1}{\alpha_0^+} \sum_m \vec{\alpha}_{n-m} \cdot \vec{\alpha}_m.$$

Periodicity requires  $\vec{\alpha}_0^L = \vec{\alpha}_0^R$ ,  $\alpha_0^{R,+} = \alpha_0^{L,+}$  and for  $\alpha_0^-$

$$\sum_{m=1}^{\infty} \vec{\alpha}_{-m}^L \cdot \vec{\alpha}_m^L = \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^R \cdot \vec{\alpha}_m^R$$

Resulting mass manifestly positive:

$$M^2 = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \vec{\alpha}_{-m} \cdot \vec{\alpha}_m = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} |\vec{\alpha}_m|^2$$

Benefits: positivity, almost all constraints gone. Drawback: Lorentz symmetry not manifest.

## 4 String Quantisation

We have seen that the classical closed string is described by

- a bunch of harmonic oscillators  $\alpha_n^{L/R}$  for the string modes;
- a relativistic particle  $(x, p)$  describing the centre of mass.

Both systems coupled via Virasoro constraints.

We have two reasonable formulations:

- Covariant formulation with  $D$  oscillators  $\alpha_n^\mu$  per mode. Virasoro constraints  $L_n^R = L_n^L = 0$  and conformal symmetry.
- Light cone formulation with  $D - 2$  oscillators  $\vec{\alpha}_n$  per mode. No constraints, full Poincaré symmetry not manifest.

### 4.1 Canonical Quantisation

Derive commutation relations for variables  $x, p, \alpha_n$ .

Recall action in conformal gauge

$$S = \frac{1}{2\pi\kappa^2} \int d^2\xi \frac{1}{2} (\dot{X}^2 - X'^2)$$

Substitute closed string mode expansion (with free time)

$$X^\mu = \kappa \sum_n \beta_n^\mu(\tau) \exp(-in\sigma)$$

Obtain tower of HO's ( $\beta_0$  free particle)

$$S = \frac{1}{2} \int d\tau \sum_n \left( \dot{\beta}_n \cdot \dot{\beta}_{-n} - n^2 \beta_n \cdot \beta_{-n} \right)$$

Canonical momentum and canonical commutator:

$$\pi_n = \dot{\beta}_n, \quad [\beta_m^\mu, \pi_n^\nu] = i\eta^{\mu\nu} \delta_{m+n}.$$

Match  $X$  with previous classical solution at  $\tau = 0$

$$x^\mu = \kappa \beta_0^\mu, \quad p^\mu = \frac{\pi_0^\mu}{\kappa}, \quad \alpha_n^{L/R, \mu} = \frac{n \beta_{\mp n}^\mu}{i\sqrt{2}} + \frac{\pi_{\mp n}^\mu}{\sqrt{2}};$$

resulting commutators in original variables

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_m^{L, \mu}, \alpha_n^{L, \nu}] = [\alpha_m^{R, \mu}, \alpha_n^{R, \nu}] = m \eta^{\mu\nu} \delta_{m+n}.$$

## 4.2 States

Compose space of states from free particle and oscillators.

- Momentum eigenstates for free particle  $|q\rangle$ .
- HO vacuum  $|0\rangle$  and excitations for each mode/orientation.

Define string vacuum state  $|0; q\rangle$

$$p^\mu |0; q\rangle = q^\mu |0; q\rangle, \quad \alpha_n^{\text{L/R}, \mu} |0; q\rangle = 0 \quad \text{for } n > 0.$$

Problem: negative norm states

$$|n, \mu; q\rangle := \alpha_{-n}^\mu |0; q\rangle, \quad |n, \mu; q|^2 = \langle 0; q | \alpha_n^\mu \alpha_{-n}^\mu | 0; q \rangle = n \eta^{\mu\mu}.$$

State not allowed by Virasoro constraints. General resolution: impose Virasoro constraints.

## 4.3 Light Cone Quantisation

Continue covariant quantisation later. Fix light cone gauge instead; only physical states.

Resulting commutators lead to positive definite states

$$[\alpha_m^{\text{L}, a}, \alpha_n^{\text{L}, b}] = [\alpha_m^{\text{R}, a}, \alpha_n^{\text{R}, b}] = m \delta^{ab} \delta_{m+n}.$$

Remember classical mass and residual constraint

$$M^2 = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\text{L}} \cdot \vec{\alpha}_m^{\text{L}} = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\text{R}} \cdot \vec{\alpha}_m^{\text{R}}.$$

Operator ordering matters! A priori free to choose. Assume normal ordering plus new constants  $a^{\text{L/R}}$ :

$$M^2 = \frac{4}{\kappa^2} \left( \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\text{L}} \cdot \vec{\alpha}_m^{\text{L}} - a^{\text{L}} \right) = \frac{4}{\kappa^2} \left( \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\text{R}} \cdot \vec{\alpha}_m^{\text{R}} - a^{\text{R}} \right).$$

Combination measures string “level”

$$N := \sum_{m=1}^{\infty} \vec{\alpha}_{-m} \cdot \vec{\alpha}_m = \sum_{m=1}^{\infty} m N_m \quad \text{with } N_m := \frac{1}{m} \vec{\alpha}_{-m} \cdot \vec{\alpha}_m.$$

Mass and constraint in terms of string level

$$M^2 = \frac{4}{\kappa^2} (N^{\text{L}} - a^{\text{L}}) = \frac{4}{\kappa^2} (N^{\text{R}} - a^{\text{R}}).$$

## 4.4 String Spectrum

Mass depends on string level. Quantisation of string level  $\rightarrow$  quantisation of mass. Level matching:  $N^{\text{L}} - a^{\text{L}} = N^{\text{R}} - a^{\text{R}}$ .

Understand string states at each level; HO's.

**Vacuum State.** Define vacuum state  $|0; q\rangle$

$$\vec{\alpha}_n^{L/R}|0; q\rangle = 0 \quad \text{for } n > 0.$$

Level zero:  $N^L = N^R = 0$ . Spin zero.

For physical state:

$$a^R = a^L = a, \quad M^2 = -\frac{4a}{\kappa^2}.$$

So far so good: spin-0 particle with  $M = 2\kappa^{-1}\sqrt{-a}$ .  $a \leq 0$ ?! Spatial extent: HO wave function  $\sim \kappa$ .

**First Level.** Lowest excited state has  $N = 1$ . Level matching and  $a^L = a^R$  implies  $N^L = N^R = 1$ . One excitation  $\vec{\alpha}_{-1}$  each from left/right movers

$$|ab; q\rangle = \alpha_{-1}^{L,a} \alpha_{-1}^{R,b} |0; q\rangle.$$

$(D-2)^2$  states of mass  $M = 2\kappa^{-1}\sqrt{1-a}$ .

Spin under transverse rotations. Three combinations:

$$\begin{aligned} |(ab); q\rangle &:= |ab; q\rangle + |ba; q\rangle - \frac{2\delta_{ab}}{D-2} |cc; q\rangle, \\ |[ab]; q\rangle &:= |ab; q\rangle - |ba; q\rangle, \\ |1; q\rangle &:= |cc; q\rangle. \end{aligned}$$

Transformation properties under  $SO(D-2)$ :

state	indices	Young tab.	“spin”
$ (ab); q\rangle$	symmetric, traceless	$\square\square$	2
$ [ab]; q\rangle$	anti-symmetric	$\square$	1
$ 1; q\rangle$	singlet	$\bullet$	0

Stabiliser (little group) for massive particle is  $SO(D-1)$ . Can fit these  $SO(D-2)$  reps. into  $SO(D-1)$  reps.? No!

Only way out: massless particle; stabiliser  $SO(D-2)$ . set  $a = a^R = a^L = 1$ .

Three types of particles:

- $|(\mathbf{ab}); q\rangle$ : massless spin-2 field. okay as free field.  
Weinberg–Witten: interactions are forbidden. except for gravitational interactions: graviton!
- $|[\mathbf{ab}]; q\rangle$ : massless 2-form field (Kalb–Ramond).  
 $B_{\mu\nu}$  with 1-form gauge symmetry  $\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$ .
- $|1; q\rangle$ : massless scalar particle (dilaton).  
different from string vacuum  $|0; q\rangle$ .

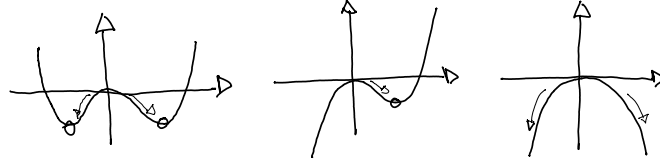
What we have learned:

- Interacting string theory includes gravity!  $\kappa$  is Planck scale.
- Graviton plus massless 2-form and scalar particles. Spatial extent  $\sim \kappa$ ; practically point-like.
- $a = a^R = a^L = 1$ .

**Tachyon.** Revisit string vacuum  $|q, 0\rangle$ :  $M^2 = -4/\kappa^2 < 0$ . Tachyon!

Problem? Not really, compare Goldstone/Higgs mechanism:

- Unstable vacuum at local maximum of a potential.
- Physical ground state at local minimum. No tachyon!
- Unclear if minimum exists. Where? What properties?



- Let us ignore. Indeed tachyon absent for superstrings!

**Higher Levels.** Levels zero and one work out. what about higher levels?

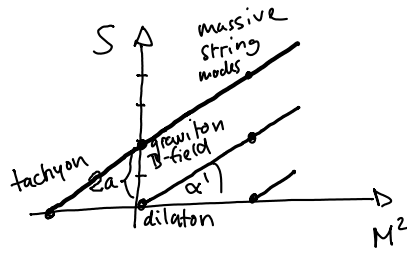
level	excitations	$SO(D-2)$	$SO(D-1)$
0	$\cdot$	$\bullet$	$\bullet$
1	$\alpha_{-1}^a$	$\square$	$?$
2	$\alpha_{-1}^a \alpha_{-1}^b$ $\alpha_{-2}^a$	$\square\square + \bullet$ $\square$	$\square\square$
3	$\alpha_{-1}^a \alpha_{-1}^b \alpha_{-1}^c$ $\alpha_{-1}^a \alpha_{-2}^b$ $\alpha_{-3}^a$	$\square\square\square + \square$ $\square\square + \square + \bullet$ $\square$	$\square\square\square$ $\square$
4	$\alpha_{-1}^a \alpha_{-1}^b \alpha_{-1}^c \alpha_{-1}^d$ $\alpha_{-1}^a \alpha_{-1}^b \alpha_{-2}^c$ $\alpha_{-2}^a \alpha_{-2}^b$ $\alpha_{-3}^a \alpha_{-1}^b$ $\alpha_{-4}^a$	$\square\square\square\square + \square\square + \bullet$ $\square\square\square + \square\square + \square + \square$ $\square\square + \bullet$ $\square\square + \square + \bullet$ $\square$	$\square\square\square\square$ $\square\square$ $\square\square$ $\bullet$
...	...	...	...

All higher levels combine into proper  $SO(D-1)$  reps.. Level matching: need to square reps..

- String describes collection of infinitely many particle types.
- Various vibration modes might correspond to elementary particles. Including gravity.
- intrinsic particle extent  $\kappa$ .  $\kappa$  is Planck scale  $\ll$  observed: Point-like particles!
- few light particles; all others at Planck mass; 1 tachyon?!
- very high excitations are long strings. mostly classical behaviour.  $M \gg 1/\kappa$  superheavy.

**Regge Trajectories.** Maximum spin (symmetric indices) increases linearly with level  $S = 2N$

$$M^2 = \frac{2S - 4a}{\kappa^2},$$



called leading Regge trajectory:

- $\alpha' = \kappa^2$  is Regge slope.
- $2a$  is Regge intercept, spin of massless particle.

Subleading trajectories for lower spins (indices anti-symmetric and in trace).

Qualitative similarity to hadron spectrum:

- Regge trajectories for hadronic resonances observed.
- $1/\kappa$  is QCD scale  $\simeq 1$  GeV
- Intercept  $a \approx -\frac{1}{2}$  rather than  $a = 1$ .
- another problem, see later.
- qualitative description of QCD flux tubes.

## 4.5 Anomalies

In light cone gauge we have broken manifest  $SO(D - 1, 1)$  Lorentz symmetry to a  $SO(D - 2)$  subgroup.

- Consequently spectrum of quantum strings organises manifestly into  $SO(D - 2)$  multiplets.
- Almost all multiplets fit into  $SO(D - 1)$  multiplets.
- Mass assignments fill Poincaré multiplets for  $a^L = a^R = 1$ .
- Poincaré symmetry broken unless  $a^L = a^R = 1$ .

Anomaly: Failure of classical symmetry in quantum theory.

Sometimes okay, not here, want strings to propagate on Minkowski background with intact Poincaré symmetry.

So far only counting, more severe problem in algebra. Commutator  $[M^{-a}, M^{-b}]$  receives contributions from  $[\alpha^-, \alpha^a]$ :

$$[M^{-a}, M^{-b}] = \sum_{n=1}^{\infty} \left( \left( \frac{D-2}{24} - 1 \right) n + \left( a - \frac{D-2}{24} \right) \frac{1}{n} \right) \dots$$

vanishes if and only if  $D = 26$  and  $a = 1$ . String theory predicts twenty-six spacetime dimensions.

Shortcut derivation: reconsider nature of intercept  $a$ .  $a$  is sum of HO ground state energies  $\frac{1}{2}\omega_n = \frac{1}{2}n$

$$a = - \sum_{n=1}^{\infty} (D-2) \frac{1}{2} \omega_n = -\frac{1}{2}(D-2) \sum_{n=1}^{\infty} n.$$

Sum divergent, black magic helps:  $\zeta$ -function regularisation

$$\zeta(x) := \sum_{k=1}^{\infty} \frac{1}{k^x}, \quad \text{i.e.} \quad a = -\frac{1}{2}(D-2)\zeta(-1) = \frac{D-2}{24}.$$

Analytical continuation  $\zeta(-1) = -\frac{1}{12}$  and  $a = 1$  predicts  $D = 26$ ! Murky derivation yields correct prediction.

## 4.6 Covariant Quantisation

In LC gauge Poincaré symmetry is subject to anomaly, but can also keep Poincaré manifest: Covariant quantisation. See how spectrum arises in covariant approach. Consider only L or R oscillators for simplicity.

**Vacuum State.**  $|0; q\rangle$  defined as before. Satisfies

$$L_{n>0}|0; q\rangle = 0 \quad \text{and} \quad L_0|0; q\rangle = \frac{\kappa^2 q^2}{4} |0; q\rangle.$$

State not annihilated by negative Virasoro modes. Instead  $\langle 0; q|L_{n<0} = 0$  hence  $\langle 0; q|(L_n - \delta_n a)|0; q\rangle = 0$ .

Impose Virasoro constraints for physical states  $|\Psi\rangle$

$$L_{n>0}|\Psi\rangle = 0, \quad L_0|\Psi\rangle = a|\Psi\rangle, \quad \langle \Psi|(L_n - \delta_n a)|\Psi\rangle = 0.$$

**One Excitation.** Generic ansatz for state

$$|\psi; q\rangle := \psi \cdot \alpha_{-1}|0; q\rangle.$$

Norm  $\bar{\psi} \cdot \psi$  potentially negative. Virasoro constraint implies

$$L_1|\psi; q\rangle = \alpha_1 \cdot \alpha_0|\psi; q\rangle = \frac{\kappa(\psi \cdot q)}{\sqrt{2}} |0; q\rangle = 0.$$

Furthermore  $L_0 = a = 1$  implies  $q^2 = 0$ . Then  $q \cdot \psi = 0$  removes negative norm state(s). Remains:

- $D - 2$  states with positive norm.
- State with  $\psi = q$  is null. Does not contribute to physics.

**Two Excitations.** Generic ansatz

$$|\phi, \psi; q\rangle := \phi_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu |0; q\rangle + \psi_\mu \alpha_{-2}^\mu |0; q\rangle.$$

Impose constraints  $L_0, L_1, L_2 = 0$  to fix  $q^2, \psi, \text{tr } \phi$ .

Remains:  $\square\square, \square, \bullet$  of  $SO(D-1)$ .

- State  $\square\square$  is positive definite.
- Ansatz for  $\square$ :  $\phi_{\mu\nu} = q_\mu \rho_\nu + \rho_\mu q_\nu$  with  $q \cdot \rho = 0$ . Negative norm for  $1 < a < 2$ . Null state for  $a = 1$ !
- Ansatz for  $\bullet$ :  $\phi_{\mu\nu} = q_\mu q_\nu + \eta_{\mu\nu} \sigma$ . Negative norm for  $D < 1$  or  $D > 26$ . Null state for  $D = 26$ !

**Virasoro Algebra.** Algebra of quantum charges  $L_n$  reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}.$$

Latter term is central charge of Virasoro,  $c = D$ . We are interested in primary states

$$L_{n>0}|\Psi\rangle = 0, \quad L_0|\Psi\rangle = a|\Psi\rangle.$$

Can apply representation theory of Virasoro algebra  $\Rightarrow$  CFT.

Proper treatment (BRST) includes ghosts. Classical conformal algebra when  $D = 26$  and  $a = 1$

$$[L_m, L_n] = (m-n)L_{m+n}.$$

Here: Conformal algebra is anomalous. Anomaly shifted to Lorentz algebra in light cone gauge.



## 5 Compactification

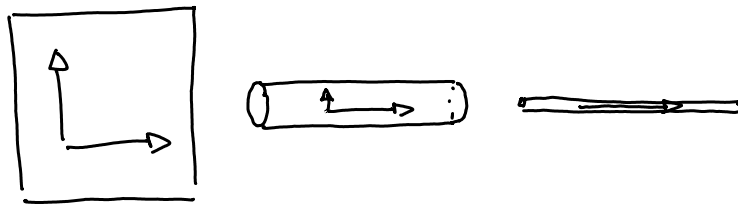
We have seen that the closed string spectrum contains:

- 1 tachyonic scalar particle (wrong vacuum).
- massless gravitons and few other particles.
- tower of particles of increasing mass (inaccessible).

But:  $D = 26$  dimensions, way too many! Gauss law: gravitational force:  
 $F \sim 1/A \sim 1/r^{24}$  not  $1/r^2$ .

### 5.1 Kaluza–Klein Modes

Idea: Compactify 22 dimensions to microscopic size. Large distances only for 4 remaining dimensions. Small compact dimensions almost unobservable.



Compactify one dimension to a circle of radius  $R$

$$X^{25} \equiv X^{25} + 2\pi R.$$

Quantum mechanical momentum quantised

$$P_{25} = \frac{n}{R}.$$

Effectively tower of massive particles  $M_{25}^2 = M_{26}^2 + n^2/R^2$ :

- Zero mode  $n = 0$  has original mass. Massless mode observable.
- Higher modes are massive,  $M \simeq 1/R$ . For very small  $R$ : practically unobservable.

Low-energy physics can be effectively four-dimensional.

### 5.2 Winding Modes

Peculiarity of strings on compact spaces: Winding.



Consider again one compact direction  $X^{25} =: X \equiv X + 2\pi R$ . Need to relax periodicity:  $X(\sigma + 2\pi) = X(\sigma) + 2\pi Rm$ .

$$X_{L/R} = \frac{1}{2}x + \frac{1}{2}\kappa^2 \left( \frac{n}{R} \mp \frac{mR}{\kappa^2} \right) \xi^{L/R} + \text{modes.}$$

Mass (for propagation in 25 non-compact dimensions)

$$M^2 = \frac{4}{\kappa^2}(N^{L/R} - a) + \left( \frac{n}{R} \mp \frac{mR}{\kappa^2} \right)^2.$$

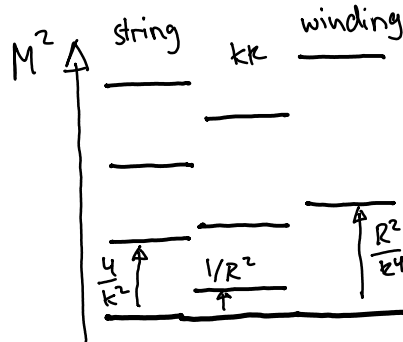
Level matching condition modified by winding

$$N^L - N^R = nm.$$

L/R average formula for mass

$$M^2 = \frac{2}{\kappa^2}(N^L + N^R - 2a) + \frac{n^2}{R^2} + \frac{m^2 R^2}{\kappa^4}.$$

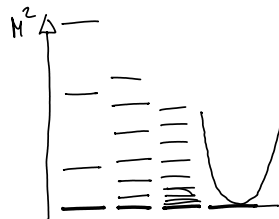
Winding also contributes mass. To hide infinitely many modes:  $\kappa$ ,  $R$  and  $\kappa^2/R$  small!



“Decompactify” circle as  $R \rightarrow \infty$ :

- Winding modes become very heavy.
- KK modes form become light and continuum.

Note: Also modes with  $N^L \neq N^R$  exist (new representations). Additional modes become infinitely heavy at  $R \rightarrow \infty$ .



Can also try to compactify circle  $R \rightarrow 0$ .

- KK modes become very heavy.
- Winding modes become light and form continuum.

Same as for  $R \rightarrow \infty$  with role of  $m$  and  $n$  interchanged. Observe: spectrum the same for  $R$  and  $\kappa^2/R$ .

Additional dimension remains observable at  $R \rightarrow 0$ ! Different from regular point particle with KK only.

### 5.3 T-Duality

Duality between small and large compactification radius. Can show at Lagrangian level: **T-duality**.

Start with action of 25-direction in conformal gauge

$$\frac{1}{2\pi\kappa^2} \int d^2\xi \frac{1}{2} \eta^{ab} \partial_a X \partial_b X$$

Action has global shift symmetry  $X \rightarrow X + \epsilon$ . For winding we would need local shift, let us make the symmetry local (“gauge”),  $A_a \rightarrow A_a - \partial_a \epsilon$

$$\frac{1}{2\pi\kappa^2} \int d^2\xi \left( \frac{1}{2} \eta^{ab} (\partial_a X + A_a) (\partial_b X + A_b) - \varepsilon^{ab} \tilde{X} \partial_a A_b \right).$$

Added two d.o.f. in  $A_a$  and one local redundancy. Remove further d.o.f. by demanding  $F_{ab} = \partial_a A_b - \partial_b A_a = 0$ . through Lagrange multiplier  $\tilde{X}$ . Done nothing (e.g.  $A_a = 0$ ).

Field  $A_a$  is algebraic, integrate out: E.o.m.

$$A_a = -\partial_a X + \eta_{ac} \varepsilon^{cb} \partial_b \tilde{X}$$

Substitute and obtain (up to boundary term)

$$\frac{1}{2\pi\kappa^2} \int d^2\xi \frac{1}{2} \eta^{ab} \partial_a \tilde{X} \partial_b \tilde{X}.$$

Same as before, but with  $\tilde{X}$  instead of  $X$ .

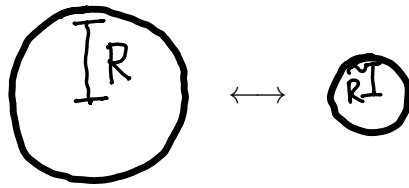
Now can set  $A_a = 0$  and obtain the duality relation

$$\partial_a X = \eta_{ac} \varepsilon^{cb} \partial_b \tilde{X}, \quad \text{i.e.} \quad \dot{X} = \tilde{X}', \quad X' = \dot{\tilde{X}}.$$

For the standard solution  $X$  we find the dual  $\tilde{X}$

$$\begin{aligned} X &= x + \kappa^2 \frac{n}{R} \tau + mR\sigma + \text{modes.} \\ \tilde{X} &= \tilde{x} + mR\tau + \kappa^2 \frac{n}{R} \sigma + \text{modes.} \end{aligned}$$

Duality interchanges  $R \leftrightarrow \tilde{R} = \kappa^2/R$  and  $m \leftrightarrow n$ .



Effectively  $R = \kappa$  is minimum compactification radius. It is indeed a special “self-dual” point. Duality between two models turns into enhanced symmetry.  $R = \kappa$  is minimum length scale in string theory: quantisation of spacetime in quantum gravity.

## 5.4 General Compactifications

So far compactified one dimension: Only circle or interval. Many choices and parameters for higher compactifications.

- sphere  $S^n$ ,
- product of spheres  $S^a \times S^{n-a}$ , different radii,
- torus  $T^n$ ,  $3n - 3$  moduli (radii, tilts),
- other compact manifolds.

Low-lying modes determined by manifold (bell).

- Compactification determines observable spectrum.
- Goal: find correct manifold to describe SM.
- Massless modes correspond to gauge symmetries.
- Superstrings: CY 3-fold preserved 1 susy.

## 6 Open Strings and D-Branes

So far we have discussed closed strings. The alternative choice is open boundary conditions.

### 6.1 Neumann Boundary Conditions

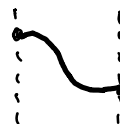
Conventionally  $0 \leq \sigma \leq \pi$  and we have to discuss  $\sigma = 0, \pi$ . Start again in conformal gauge

$$S = \frac{1}{2\pi\alpha'^2} \int d^2\xi \frac{1}{2} \eta^{ab} \partial_a X \cdot \partial_b X.$$

Variation including boundary terms due to partial integration

$$\begin{aligned} \delta S &= \frac{1}{2\pi\alpha'^2} \int d^2\xi \eta^{ab} \partial_a \delta X \cdot \partial_b X \\ &= \frac{1}{2\pi\alpha'^2} \int d^2\xi \partial_a (\eta^{ab} \delta X \cdot \partial_b X) - \dots \\ &= \frac{1}{2\pi\alpha'^2} \int d\tau (\delta X(\pi) \cdot X'(\pi) - \delta X(0) \cdot X'(0)) - \dots \end{aligned}$$

Boundary e.o.m. imply Neumann conditions (alternative later)

$$X'(0) = X'(\pi) = 0.$$


Virasoro constraints

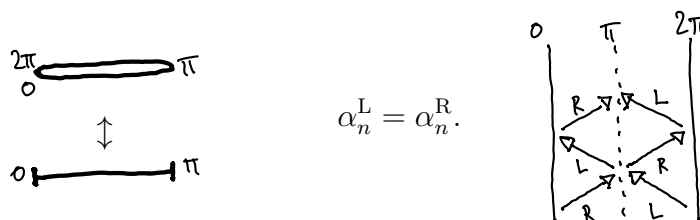
$$X' \cdot \dot{X} = X'^2 + \dot{X}^2 = 0$$

imply that end points move at speed of light  $\dot{X}^2 = 0$ . (no free ends of analogous soap film:  $\dot{X}^2 = 0$  implies  $\dot{X} = 0$ .)

### 6.2 Solutions and Spectrum

Same equations in the string bulk, recycle solution.

**Doubling Trick.** Map two copies of open string to closed string twice as long:  $\sigma \equiv 2\pi - \sigma$ . Gluing condition  $X' = 0$  at  $\sigma = 0, \pi$  implies



Left movers are reflected into right movers at boundary. One copy of oscillators and Virasoro algebra

$$X^\mu = x^\mu + 2\kappa^2 p^\mu \tau + \sum_{n \neq 0} \frac{i\kappa}{\sqrt{2n}} \alpha_n^\mu (\exp(-in\xi^L) + \exp(-in\xi^R)).$$

(momentum  $p$  is doubled because  $\sigma$  integration is halved)

**Quantisation.** Analogous to closed strings. Same anomaly conditions  $D = 26$ ,  $a = 1$  (from bulk). Resulting spectrum (note different prefactor due to  $p$ ).

$$M^2 = \frac{1}{\kappa^2} (N - a).$$

Only single copy of oscillators at each level.

- level 0: singlet tachyon (of half “mass”).
- level 1: massless vector: Maxwell field.
- level 2: massive spin-2 field  $\square\square$ .
- ...

Same as discussion for closed string without squaring!

Massless modes are associated to local symmetries:

- of open string are spin-1 gauge fields,
- of closed string are spin-2 gravitation fields.

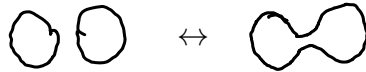
**String Interactions.** Open and closed strings interact:

- Two ends of string can join.



Open strings must include closed strings. Different “vacuum” states  $|0; q\rangle_c$  and  $|0; q\rangle_o$  in same theory.

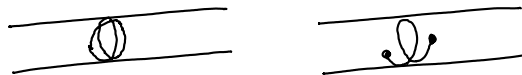
- Opening of string can be suppressed. Closed string can live on their own.



String theory always contains gravity; May or may not include gauge field(s).

### 6.3 Dirichlet Boundary Conditions

Now consider compactification for open strings. Almost the same as for closed string. No winding modes because open string can unwind.



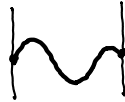
**T-Duality.** What about applying T-duality? Introduce dual fields

$$X' = \dot{\tilde{X}}, \quad \dot{X} = \tilde{X}'.$$

Boundary conditions translate to

$$X'^{\mu} = 0 \quad (\mu = 0, \dots, 24), \quad \dot{\tilde{X}}^{25} = 0.$$

Dirichlet boundary condition for dual coordinate  $\tilde{X}^{25}$ . Corresponds to alternate choice of boundary e.o.m.  $\delta X^{25} = 0$ .



KK modes turn into winding modes:

$$\Delta \tilde{X}^{25} = \int d\sigma \tilde{X}'^{25} = \int d\sigma \dot{X}^{25} = 2\pi \kappa^2 p_{25} = \frac{2\pi \kappa^2 n}{R} = 2\pi n \tilde{R}.$$

Strings start and end at same  $x_0 \equiv x_0 + 2\pi \tilde{R}$ . Note: No momentum  $\tilde{P}_{25}$  because position fixed. Role of KK and winding exchanged.

Dirichlet condition modifies oscillator relation:

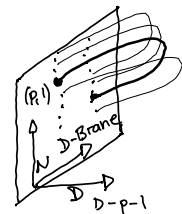
$$\alpha_n^{L,25} = -\alpha_n^{R,25}.$$

Although Dirichlet condition  $\tilde{X}^{25} = \text{const.}$  appears unnatural, it has to be part of string theory (on compact spaces).

**D-Branes.** Take seriously.

At boundary can choose:

- Neumann condition  $X'^{\mu} = 0$  or
- Dirichlet condition  $X^{\mu} = \text{fixed}$  for each direction  $\mu$  individually.



Geometrical picture: String ends confined to  $Dp$ -branes.

- $p + 1$  dimensional  $(p, 1)$  submanifolds of spacetime.
- Dirichlet conditions for  $D - p - 1$  orthogonal directions.
- Neumann conditions for  $p + 1$  parallel directions.
- D-branes can be curved (normal depends on position).

T-duality maps between  $Dp$  and  $D(p \pm 1)$  branes.

Pure Neumann conditions are spacetime-filling D-brane.

Strings propagate on backgrounds with D-branes:

- spacetime bulk curvature governs string bulk propagation,
- D-branes govern string end propagation.

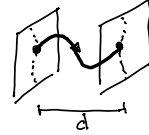
Even more: Will continue discussion later.

## 6.4 Multiple Branes

Can have multiple branes of diverse types. Open strings stretch between two branes.

**Parallel Branes.** Simplest case: Two parallel planar  $Dp$ -branes located at  $X^{25} = 0, d$  in non-compact Minkowski space

$$X^\mu = 2\kappa^2 p^\mu \tau + \text{modes}, \quad X^{25} = \frac{\sigma d}{\pi} + \text{modes}.$$



Resulting (quantum) mass spectrum in  $p + 1$  dimensions

$$M^2 = \frac{d^2}{4\pi^2\kappa^4} + \frac{1}{\kappa^2} (N - a).$$

- Spin-1 particle at level-1 with mass  $M = d/2\pi\kappa^2$ .
- Vector massless at coincident branes.
- Tachyon for  $d < 2\pi\kappa$ : Instability for nearby D-branes.

**Multiple Branes.** Consider now  $N$  parallel branes.

There are  $N^2$  types of open string (and 1 closed): String vacua distinguished by Chan-Paton factors

$$|0; q; a\bar{b}\rangle_o, \quad a, \bar{b} = 1, \dots, N.$$

with general mass formula

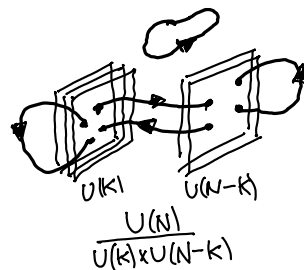
$$M_{a\bar{b}}^2 = \frac{d_{a\bar{b}}^2}{4\pi^2\kappa^4} + \frac{1}{\kappa^2} (N - a).$$

Consider vector particles at level 1 with mass  $d_{a\bar{b}}/2\pi\kappa^2$ .

- Always  $N$  massless vectors. Gauge symmetry:  $U(1)^N$ .
- $K$  coincident branes contribute  $K^2$  massless vectors. Enhanced gauge symmetry  $U(1)^K \rightarrow U(K)$ .
- Massive vectors indicate spontaneously broken symmetries.

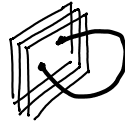
Geometric picture of gauge symmetries:

- Stack of  $N$  branes have local  $U(N)$  symmetry.
- Separating branes breaks symmetry to  $U(K) \times U(N - K)$ .
- Creates  $2K(N - K)$  massive vectors.





Can also produce  $SO(N)$  and  $Sp(N)$  symmetries: Unoriented strings, strings on orientifolds (spacetime involution paired with orientation reversal).



**Brane Worlds.** Can design many different situations.

Combine:

- non-compact dimensions,
- D-branes,
- intersections of D-branes and non-compact dimensions,
- orientifold action.

Consider physics:

- along non-compact dimensions,
- within D-branes.

Qualitative features:

- Massless vectors indicate gauge symmetries.
- Light vectors indicate spontaneous symmetry breaking.
- Tachyons indicate instabilities of D-branes or spacetime.

String theory becomes framework analogous to QFT:

- D-brane arrangements and compact directions (discrete),
- moduli for D-branes and non-compact spaces (continuous).

Physics: Try to design the standard model at low energies.

Mathematics: Dualities relate various situations.

## 7 Conformal Field Theory

So far considered mostly string spectrum:

- equations of motion (local),
- closed/open periodicity conditions (global),
- quantisation.

Quantum mechanics of infinite tower of string modes  $\alpha_n$ .

Next will consider local picture on worldsheet: Fields  $X(\xi)$ . Quantisation  $\rightarrow$  Quantum Field Theory (QFT). Will need for string scattering.

Reparametrisation invariance:

- worldsheet coordinates  $\xi$  artificial,
- gauge fixing: conformal gauge,
- worldsheet coordinates  $\xi$  meaningful,
- diffeomorphisms  $\rightarrow$  residual conformal symmetry,
- Conformal Field Theory (CFT).

CFT: QFT making use of conformal symmetry.

- do not calculate blindly,
- structure of final results dictated by symmetry,
- conformal symmetry: large amount, exploit!

Let us scrutinise conformal symmetry:

- Central framework in string theory,
- but also useful for many 2D statistical mechanics systems.

### 7.1 Conformal Transformations

Special coordinate transformation:

- all angles unchanged,
- definition of length can change,

Metric preserved up to scale

$$g'_{\mu'\nu'}(x') = \frac{dx^\mu}{dx^{\mu'}} \frac{dx^\nu}{dx^{\nu'}} g_{\mu\nu}(x) \stackrel{!}{=} f(x) g_{\mu'\nu'}(x)$$

**Action on Coordinates.** Generally in  $D$  dimensions

- Lorentz rotations  $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$ ,
- translations  $x^\mu \rightarrow x^\mu + a^\mu$ ,
- scale transformations / dila(ta)tions  $x^\mu \rightarrow s x^\mu$ ,
- conformal inversions (discrete)  $x^\mu \rightarrow x^\mu/x^2$ ,

- conformal boosts (inversion, translation, inversion).
- Conformal group:  $SO(D, 2)$  (rather: universal cover).

**Action on Fields.** E.g. a free scalar

$$S \sim \int d^D x \frac{1}{2} \partial_\mu \Phi(x) \partial^\mu \Phi(x).$$

- Manifest invariance under Lorentz rotations & translations

$$\Phi'(x) = \Phi(\Lambda x + a).$$

- Invariance under scaling  $x' = sx$  requires

$$\Phi'(x) = s^{(D-2)/2} \Phi(sx).$$

- Invariance under inversions

$$\Phi'(x) = (x^2)^{-(D-2)/2} \Phi(1/x).$$

Similar (but more complicated) rules for:

- scalar field  $\phi(x)$  with different scaling  $\phi'(x) = s^\Delta \phi(sx)$ ,
- spinning fields  $\rho_\mu, \dots$ ,
- derivatives  $\partial_\mu \Phi, \partial_\mu \partial_\nu \Phi, \partial^2 \Phi, \dots$

**2D Conformal Symmetries.** QFT's in 2D are rather tractable. CFT's in 2D are especially simple:

- Conformal group splits  $SO(2, 2) \simeq SL(2, \mathbb{R})_L \times SL(2, \mathbb{R})_R$
- $SL(2, \mathbb{R})_{L/R}$  act on coordinates as (drop L/R)

$$\xi' = \frac{a\xi + b}{c\xi + d}, \quad \delta\xi = \beta + \alpha\xi - \gamma\xi^2;$$

$\beta^{L/R}$  are two translations,  $\alpha^{L/R}$  are rotations and scaling,  $\gamma^{L/R}$  are two conformal boosts.

- $SL(2, \mathbb{R})_{L/R}$  extends to infinite-dimensional Virasoro

$$\delta\xi^{L/R} = \epsilon^{L/R}(\xi^{L/R}) = \sum_n \epsilon_n^{L/R} (\xi^{L/R})^{1-n}.$$

- Boundaries typically distorted by Virasoro. Only subalgebra preserves boundaries, e.g.  $SL(2, \mathbb{R})$ .

## 7.2 Conformal Correlators

In a quantum theory interested in

- spectrum of operators (string spectrum),

- probabilities,
- expectation value of operators on states.

In QFT compute (vacuum) expectation values:

- momentum eigenstates: particle scattering, S-matrix

$$\langle \vec{q}_1, \vec{q}_2, \dots | S | \vec{p}_1, \vec{p}_2, \dots \rangle = \langle 0 | a(\vec{q}_1) a(\vec{q}_2) \dots S \dots a^\dagger(\vec{p}_2) a^\dagger(\vec{p}_1) | 0 \rangle$$

- position eigenstates: time-ordered correlation functions

$$\langle \Phi(x_1) \Phi(x_2) \dots \rangle = \langle 0 | T[\Phi(x_1) \Phi(x_2) \dots] | 0 \rangle$$

**Correlator of String Coordinates.** Can compute a worldsheet correlator using underlying oscillator relations

$$\begin{aligned} \langle 0 | X^\nu(\xi_2) X^\mu(\xi_1) | 0 \rangle &= -\frac{\kappa^2}{2} \eta^{\mu\nu} \log(\exp(i\xi_2^L) - \exp(i\xi_1^L)) \\ &\quad - \frac{\kappa^2}{2} \eta^{\mu\nu} \log(\exp(i\xi_2^R) - \exp(i\xi_1^R)) + \dots \end{aligned}$$

Can reproduce from CFT? Scalar  $\phi$  of dimension  $\Delta$ :

$$\langle \phi(x_1) \phi(x_2) \rangle = F(x_1, x_2)$$

Correlator should be invariant!

- Translation invariance

$$F(x_1, x_2) = F(x_1 - x_2) =: F(x_{12}).$$

Just one vector variable.

- Invariance under Lorentz rotations

$$F(x_{12}) = F(x_{12}^2).$$

Just a scalar variable.

- Scaling invariance

$$\langle \phi(x_1) \phi(x_2) \rangle \stackrel{!}{=} \langle \phi'(x_1) \phi'(x_2) \rangle = s^{2\Delta} \langle \phi(sx_1) \phi(sx_2) \rangle,$$

hence  $F(x_{12}^2) = s^{2\Delta} F(s^2 x_{12}^2)$  and

$$F(x_{12}^2) = \frac{N}{(x_{12}^2)^\Delta}.$$

Just a (normalisation) constant  $N$ !

**Logarithmic Correlator.** Our scalar has scaling dimension  $\Delta = (D - 2)/2 = 0$ . Constant correlator  $F(x_1, x_2) = N$ ?! Not quite: Take limit  $D = 2 + 2\epsilon$ ,  $N = N_2/\epsilon$

$$F(x_1, x_2) = \frac{N_2}{\epsilon(x_{12}^2)^\epsilon} \rightarrow \frac{N_2}{\epsilon} - N_2 \log x_{12}^2 + \dots$$

Note:  $\Delta = 0$  correlator can be logarithmic. Still not there. Use LC coordinates  $x_{12}^2 = -x_{12}^L x_{12}^R$  and identify

$$x^L = \exp(i\xi^L), \quad x^R = \exp(i\xi^R).$$

Why the identification?

- 2D conformal transformation,
- closed string periodicity  $\sigma \equiv \sigma + 2\pi$ , but  $x^{L/R}$  unique!
- choose appropriate coordinates for boundaries.

String coordinates are functions of  $x^{L/R}$  except for linear dependence on  $\tau = -\frac{i}{2} \log(x^L x^R)$ . Better choice of field  $\partial X^\mu / \partial x^{L/R}$ :

$$\langle 0 | \partial_L X^\nu(x_2) \partial_L X^\mu(x_1) | 0 \rangle = \frac{-\frac{1}{2} \kappa^2 \eta^{\mu\nu}}{(x_2^L - x_1^L)^2}$$

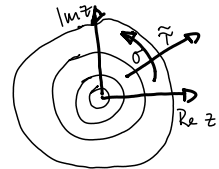
More manifestly conformal!

**Wick Rotation.** In this context: Typically perform Wick rotation  $\tau = -i\tilde{\tau}$  (now  $\tilde{\tau}$  real)

$$\exp(i\xi^L) = \exp(\tilde{\tau} - i\sigma) =: \bar{z}, \quad \exp(i\xi^R) = \exp(\tilde{\tau} + i\sigma) =: z.$$

Cylindrical coordinates for (euclidean) string:

- radius  $|z|$  is exponential euclidean time  $\tilde{\tau}$ ,
- $\sigma$  is angular coordinate (naturally periodic).



Standard treatment: Euclidean CFT

- Worldsheet coordinates  $z$  and  $\bar{z}$  are complex conjugates.
- Fields are functions  $f(z, \bar{z})$  of complex  $z$ .
- String coordinates are holomorphic functions

$$X(z, \bar{z}) = X(z) + \bar{X}(\bar{z}).$$

- Conformal transformations are holomorphic.
- Employ powerful functional analysis: residue theorems.

Euclidean WS convenient and conventional. Could as well work on Minkowski worldsheet, nothing lost!

### 7.3 Local Operators

We understand the basic string coordinate field  $X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$ , or better  $\partial X(z)$  and  $\bar{\partial} \bar{X}(\bar{z})$ .

Basic objects in a CFT are local operators  $\mathcal{O}_i(z, \bar{z})$ :

- products of fields  $X$  and derivatives  $\partial^n \bar{\partial}^{\bar{n}} X$ ,
- evaluated at the same point  $(z, \bar{z})$  on the worldsheet,
- normal ordered  $\mathcal{O}_i = : \dots :$  implicit (no self-correlations),
- for example  $\mathcal{O}_1 = :(\partial X)^2:$ ,  $\mathcal{O}_2^{\mu\nu} = :X^\mu \partial X^\nu:$   $- :X^\nu \partial X^\mu:$ ,  $\dots$

Local operators behave

- classically as the sum of constituents,
- quantum-mechanically as independent entities: recall quantum effects in Virasoro charges  $(\partial X)^2$ !

Main task: classify local operators.

**Descendants.** All local operators transform under shifts  $(\delta z, \delta \bar{z}) = (\epsilon, \bar{\epsilon})$  as

$$\delta \mathcal{O} = \epsilon \partial \mathcal{O} + \bar{\epsilon} \bar{\partial} \mathcal{O}.$$

An operator  $\partial^n \bar{\partial}^{\bar{n}} \mathcal{O}$  is called a descendant of  $\mathcal{O}$ . Shifts are symmetries: No need to consider descendants.

**Weights.** Most local operators classified by weights  $(h, \bar{h})$ . Transformation under  $(z, \bar{z}) \rightarrow (sz, \bar{s}\bar{z})$  or  $\delta(z, \bar{z}) = (\epsilon z, \bar{\epsilon} \bar{z})$

$$\begin{aligned} \mathcal{O}'(z, \bar{z}) &= s^h \bar{s}^{\bar{h}} \mathcal{O}(sz, \bar{s}\bar{z}), \\ \delta \mathcal{O} &= \epsilon(h\mathcal{O} + z\partial\mathcal{O}) + \bar{\epsilon}(\bar{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}). \end{aligned}$$

Transformations are scaling and rotation, hence scaling dimension  $\Delta = h + \bar{h}$  and spin  $S = h - \bar{h}$ .

For unitary CFT: Both  $h, \bar{h}$  are real and non-negative. E.g. weights:  $\partial X \rightarrow (1, 0)$ ,  $(\partial X)^2 \rightarrow (2, 0)$ .

Products of local operators  $\mathcal{O} = \mathcal{O}_1 \mathcal{O}_2$ :

- total weight is sum of individual weights classically;
- weights usually not additive in quantum theory!

Note:  $X$  does not have proper weights, but  $\partial X$  does.

**Quasi-Primary Operators.** A local operator with weights  $(h, \bar{h})$  is called quasi-primary if

$$\mathcal{O}'(z, \bar{z}) = \left( \frac{dz'}{dz} \right)^h \left( \frac{d\bar{z}'}{d\bar{z}} \right)^{\bar{h}} \mathcal{O}(z', \bar{z}').$$

for all  $SL(2, \mathbb{C})$  Möbius transformations

$$z' = \frac{az + b}{cz + d}, \quad \bar{z}' = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}.$$

For infinitesimal boosts  $\delta(z, \bar{z}) = (\epsilon z^2, \bar{\epsilon} \bar{z}^2)$  it must satisfy

$$\delta \mathcal{O} = \epsilon(2hz\mathcal{O} + z^2\partial\mathcal{O}) + \bar{\epsilon}(2\bar{h}\bar{z}\mathcal{O} + \bar{z}^2\bar{\partial}\mathcal{O}).$$

Descendants of quasi-primaries are not quasi-primary.

Need to consider only quasi-primary operators.

**Primary Operators.** An operator is called primary if it satisfies the quasi-primary conditions for all transformations

$$(z, \bar{z}) \rightarrow (z'(z), \bar{z}'(\bar{z})) \quad \text{or} \quad (\delta z, \delta \bar{z}) = (\zeta(z), \bar{\zeta}(\bar{z})).$$

Infinitesimally

$$\delta \mathcal{O} = (h \partial \zeta \mathcal{O} + \zeta \partial \mathcal{O}) + (\bar{h} \bar{\partial} \bar{\zeta} \mathcal{O} + \bar{\zeta} \bar{\partial} \mathcal{O}).$$

Note: Correlators are only locally invariant. Only a subclass of conformal transformations (e.g. Möbius) leaves correlators globally invariant.

**Example.** Operator  $\mathcal{O}^\mu = \partial X^\mu$  is primary;  $(h, \bar{h}) = (1, 0)$ .

$$\langle \mathcal{O}_1^\mu \mathcal{O}_2^\nu \rangle = \frac{-\frac{1}{2} \kappa^2 \eta^{\mu\nu}}{(z_1 - z_2)^2}.$$

Invariance under  $\delta z = z^{1-n}$ :

- exact for  $|n| \leq 1$  (Möbius),
- up to polynomials for  $|n| > 1$  (small w.r.t.  $1/(z_1 - z_2)^2$ ).

**State-Operator Map.** There is a one-to-one map between

- quantum states on a cylinder  $\mathbb{R} \times S^1$  and
- local operators (at  $z = 0$ ).

Consider the conformal map

$$z = \exp(+i\zeta), \quad \bar{z} = \exp(-i\bar{\zeta}), \quad \zeta, \bar{\zeta} = \sigma \mp i\tilde{\tau}.$$

State given by wave function at constant  $\tilde{\tau} = -\text{Im} \zeta$ :

- Time evolution is radial evolution in  $z$  plane.
- Asymptotic time  $\tilde{\tau} \rightarrow -\infty$  corresponds to  $z = 0$ .
- Local operator at  $z = 0$  to excite asymptotic wave function.
- Unit operator 1 corresponds to vacuum.

## 7.4 Operator Product Expansion

In a CFT we wish to compute correlation functions

$$\langle \mathcal{O}_1(\xi_1) \mathcal{O}_2(\xi_2) \dots \mathcal{O}_n(\xi_n) \rangle = F_{12\dots n}.$$

Suppose  $\xi_1 \approx \xi_2$ ; then can Taylor expand

$$\mathcal{O}_1(\xi_1) \mathcal{O}_2(\xi_2) = \sum_{n=0}^{\infty} \frac{1}{n!} (\xi_2 - \xi_1)^n \mathcal{O}_1(\xi_1) \partial^n \mathcal{O}_2(\xi_1).$$

Converts local operators at two points into a sum of local operators at a single point. Classical statement is exact.

**Quantum OPE.** Quantum-mechanically there are additional contributions from operator ordering (normal ordering implicit). Still product of local operators can be written as sum of some local operators

$$\mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2) = \sum_i C_{12}^i(\xi_2 - \xi_1)\mathcal{O}_i(\xi_1).$$

More precise formulation with any (non-local) operators “...”

$$\langle \mathcal{O}_1(\xi_1)\mathcal{O}_2(\xi_2)\dots \rangle = \sum_i C_{12}^i(\xi_2 - \xi_1)\langle \mathcal{O}_i(\xi_1)\dots \rangle.$$

This statement is called Operator Product Expansion (OPE).  $C_{ij}^k(\xi_2 - \xi_1)$  are called structure constants & conformal blocks. Sum extends over all local operators (including descendants).

Idea: Every (non-local) operator can be written as an expansion in local operators. Analog: Multipole expansion.

It works exactly in any CFT and is a central tool.

**Higher Points.** Can formally compute higher-point correlation functions:

$$F_{123\dots n} = \sum_i C_{12}^i F_{i3\dots n}$$

Apply recursively to reduce to single point.

One-point function is trivial (except for unit operator 1)

$$\langle \mathcal{O}_i \rangle = 0, \quad \langle 1 \rangle = 1.$$

Higher-point function reduced to sequence of  $C_{ij}^k$ :

- vast simplification,
- need only  $C_{ij}^k$  for correlators in CFT,
- hard to compute in practice,
- result superficially depends on OPE sequence (crossing).

**Lower Points.** Two-point function is OPE onto unity

$$F_{ij} = \langle \mathcal{O}_i \mathcal{O}_j \rangle = \sum_k C_{ij}^k \langle \mathcal{O}_k \rangle = C_{ij}^1.$$

Three-point function determines OPE constants

$$F_{ijk} = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \sum_l C_{ij}^l \langle \mathcal{O}_k \mathcal{O}_l \rangle = \sum_l F_{kl} C_{ij}^l.$$

Lower-point functions restricted by conformal symmetry:

- Two-point function only for related operators.
- No two-point or three-point conformal invariants. Can map triple of point to any other triple of points.



- Coordinate dependence of two-point function fixed

$$F_{ij} \sim \frac{N_{ij}}{|\xi_i - \xi_j|^{2\Delta_i}}.$$

Numerator  $N$  depends on dimension, spin, level of descendant and operator normalisation.

- Coordinate dependence of three-point function fixed

$$F_{ijk} \sim \frac{N_{ijk}}{|\xi_i - \xi_j|^{\Delta_{ij}} |\xi_j - \xi_k|^{\Delta_{jk}} |\xi_k - \xi_i|^{\Delta_{ki}}}$$

with scaling weights  $\Delta_{ij} = \Delta_i + \Delta_j - \Delta_k$ . Numerators  $N$  depend on dimension, spin, level of descendant and operator normalisation.

- Three-point functions exist for three different operators.

Normalise operators, then CFT data consists of

- scaling dimensions, spins: spectrum,
- coefficients of three-point function: structure constants.

## 7.5 Stress-Energy Tensor

The Noether currents for spacetime symmetries are encoded into the conserved stress-energy tensor  $T_{\alpha\beta}$

$$T_{\alpha\beta} = -\frac{1}{4\pi\kappa^2} ((\partial_\alpha X) \cdot (\partial_\beta X) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}(\partial_\gamma X) \cdot (\partial_\delta X))$$

Object of central importance for CFT/OPE! Trace is exactly zero: Weyl symmetry. Remaining components  $T_{LL}$  and  $T_{RR}$  translate to euclidean

$$T = -\frac{1}{\kappa^2}(\partial X)^2, \quad \bar{T} = -\frac{1}{\kappa^2}(\bar{\partial}\bar{X})^2.$$

Ignore string physical state condition  $T = \bar{T} = 0$ .

**Conservation.** Current  $J(z) = \zeta(z)T(z)$  for  $\delta z = \zeta(z)$ . Classical conservation  $\bar{\partial}J = 0$  by means of e.o.m.. QFT: Conservation replaced by Ward identity:

$$\bar{\partial}J(z)\mathcal{O}(w, \bar{w}) = 2\pi \delta^2(z - w, \bar{z} - \bar{w}) \delta\mathcal{O}(w, \bar{w}).$$

Current  $J$  conserved except at operator locations.

OPE: Integrate  $z$  over small ball around  $w$

$$\frac{1}{2\pi} \int_{|z-w|<\epsilon} d^2z \dots$$

Evaluate integration over  $\bar{z}$  ( $\int d^2z \bar{\partial} \dots = -i \int dz \dots$ )

$$\frac{1}{2\pi i} \int_{|z-w|=\epsilon} dz J(z)\mathcal{O}(w, \bar{w}) = \delta\mathcal{O}(w, \bar{w}).$$

Similarly for  $\bar{T}$ . Consider only holomorphic part.

**Stress-Energy OPE.** Derive OPE of  $\mathcal{O}$  and  $T$ .

First consider translation  $\delta z = \epsilon$ ,  $\delta\mathcal{O} = \epsilon\partial\mathcal{O}$ . Need simple pole to generate residue

$$T(z)\mathcal{O}(w, \bar{w}) = \dots + \frac{\partial\mathcal{O}(w, \bar{w})}{z-w} + \dots$$

Further terms with higher poles and polynomials in "...".

Suppose  $\mathcal{O}$  has holomorphic weight  $h$ . Consider scaling  $\delta z = \epsilon z$ ,  $\delta\mathcal{O} = \epsilon(h\mathcal{O} + z\partial\mathcal{O})$ . Substitute and require following poles in OPE

$$T(z)\mathcal{O}(w, \bar{w}) = \dots + \frac{h\mathcal{O}(w, \bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w, \bar{w})}{z-w} + \dots$$

Next suppose  $\mathcal{O}$  is quasi-primary. Consider scaling  $\delta z = \epsilon z^2$ ,  $\delta\mathcal{O} = \epsilon(2hz\mathcal{O} + z^2\partial\mathcal{O})$ . Substitute and require absence of cubic pole

$$T(z)\mathcal{O}(w, \bar{w}) = \dots + \frac{0}{(z-w)^3} + \frac{h\mathcal{O}(w, \bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w, \bar{w})}{z-w} + \dots$$

Finally suppose  $\mathcal{O}$  is primary. Leads to absence of higher poles

$$T(z)\mathcal{O}(w, \bar{w}) = \frac{h\mathcal{O}(w, \bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w, \bar{w})}{z-w} + \dots$$

Note that derivatives shift poles by one order.

Descendants are not (quasi-)primaries.

OPE of stress-energy tensor. Compute explicitly (Wick):

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

Result applies to general CFT's. Virasoro algebra!

- $T$  is a local operator,
- $T$  has holomorphic weight  $h = 2$  (classical),
- $T$  is quasi-primary,
- $T$  is not primary (unless  $c = 0$ ),
- quartic pole carries central charge  $c = D$ .

Conformal transformations for  $T$  almost primary:

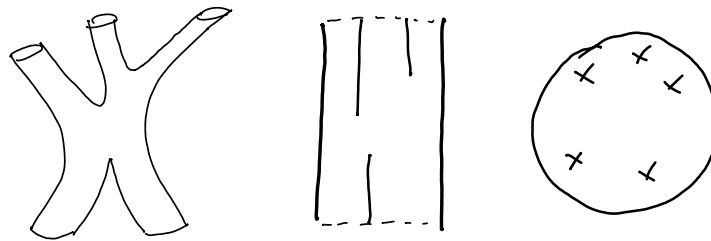
$$\begin{aligned} \delta T &= \delta z \partial T + 2 \partial \delta z T + \frac{c}{12} \partial^3 \delta z, \\ T'(z) &= \left( \frac{dz'}{dz} \right)^2 \left( T(z') + \frac{c}{12} S(z', z) \right), \\ S(z', z) &= \left( \frac{d^3 z'}{dz^3} \right) \left( \frac{dz'}{dz} \right)^{-1} - \frac{3}{2} \left( \frac{d^2 z'}{dz^2} \right)^2 \left( \frac{dz'}{dz} \right)^{-2} \end{aligned}$$

Additional term  $S$  is Schwarzian derivative. Zero for Möbius transformations.

## 8 String Scattering

Compute a string scattering amplitude. Two methods:

- worldsheet junction(s). string cylinders with cuts. integration over junctions.
- vertex operators. integration over punctures locations.



### 8.1 Vertex Operators

State-operator map:

- Which operator creates a string?
- How to specify the momentum  $q$ ?
- How to specify the string modes?

Solution is related to the operator  $\mathcal{O}[q] = :\exp(iq_\mu X^\mu):$ . Why?

- Momentum eigenstate: phase for translation  $\exp(iq_\mu \epsilon^\mu)$ .

Compute OPE with stress-energy  $T$

$$T(z)\mathcal{O}[q](w, \bar{w}) = \frac{\frac{1}{4}\kappa^2 q^2 \mathcal{O}[q](w, \bar{w})}{(z-w)^2} + \frac{\partial \mathcal{O}[q](w, \bar{w})}{z-w} + \dots$$

Primary operator with weights  $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$ !

- non-trivial, non-integer weight,
- quantum effect  $\sim \kappa^2$ .

Consider two-point correlator

$$\langle \mathcal{O}_1[q_1] \mathcal{O}_2[q_2] \rangle \simeq |z_1 - z_2|^{\kappa^2(q_1 \cdot q_2)}.$$

In fact, zero mode  $X^\mu = x^\mu + \dots$  contributes extra factor

$$\int d^D x \exp(iq_1 \cdot x + iq_2 \cdot x) \sim \delta^D(q_1 + q_2).$$

Hence compatible with primary of weight  $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$

$$\langle \mathcal{O}_1[q_1] \mathcal{O}_2[q_2] \rangle \simeq \frac{\delta^D(q_1 + q_2)}{|z_1 - z_2|^{\kappa^2 q_1^2}}.$$

Operator  $\mathcal{O}[q](z, \bar{z})$  creates a string state at  $(z, \bar{z})$ . Worldsheet location unphysical, integrate:

$$V[q] = g_s \int d^2z \mathcal{O}[q](z, \bar{z}).$$

Can only integrate weight  $(1, 1)$  primary operators. Hence:

- mass  $M^2 = -q^2 = -4/\kappa^2$ ; string tachyon!
- intercept  $a = \bar{a} = 1$  due to worldsheet integration.

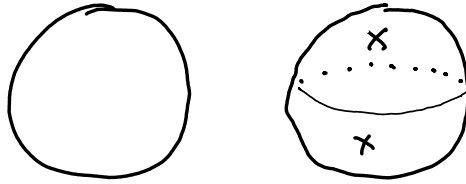
What about excited strings? Level-1 corresponds to

$$V^{\mu\nu}[q] = g_s \int d^2z \partial X^\mu \bar{\partial} X^\nu \mathcal{O}[q].$$

- weight is  $(1 + \frac{1}{4}\kappa^2 q^2, 1 + \frac{1}{4}\kappa^2 q^2) = (1, 1)$  for massless  $q$ .
- primary condition removes unphysical polarisations.
- gauge d.o.f. are total derivatives.

Vertex operator picture:

- CFT vacuum is empty worldsheet (genus 0, no punctures).
- $\int d^2z \mathcal{O}[q](z, \bar{z})$  is string vacuum  $|0; q\rangle$  (add puncture).
- $\int d^2z \dots \mathcal{O}[q](z, \bar{z})$  are excited string states. Insertions of  $\partial^n X^\mu$  correspond to string oscillators  $\alpha_n^\mu$ , insertions of  $\bar{\partial}^n X^\mu$  correspond to  $\bar{\alpha}_n^\mu$ .



## 8.2 Veneziano Amplitude

Consider  $n$ -point amplitude (with  $\mathcal{O}_k = \mathcal{O}[q_k](z_k, \bar{z}_k)$ )

$$A_n \sim \frac{1}{g_s^n} \langle V_1 \dots V_n \rangle \sim g_s^{n-2} \int d^{2n}z \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$$

- simplest to use tachyon vertex operators,
- can do others, but add complications (fields),
- computation & result qualitatively the same.

Perform Wick contractions and zero mode integration

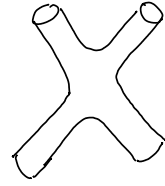
$$\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle \sim \delta^D(Q) \prod_{j < k} |z_j - z_k|^{\kappa^2 q_j \cdot q_k}.$$

Integral invariant under Möbius transformations ( $q_k^2 = 4/\kappa^2$ ). Map three punctures to fixed positions  $z_1 = \infty, z_2 = 0, z_3 = 1$ . Remaining integral for  $n = 4$  strings

$$A_4 \sim g_s^2 \delta^D(Q) \int d^2z |z|^{\kappa^2 q_2 \cdot q_4} |1 - z|^{\kappa^2 q_3 \cdot q_4}$$

can be performed

$$A_4 \sim g_s^2 \delta^D(Q) \frac{\Gamma(-1 - \kappa^2 s/4) \Gamma(-1 - \kappa^2 t/4) \Gamma(-1 - \kappa^2 u/4)}{\Gamma(+2 + \kappa^2 s/4) \Gamma(+2 + \kappa^2 t/4) \Gamma(+2 + \kappa^2 u/4)}.$$



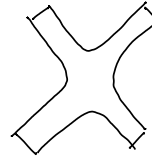
Mandelstam invariants:

$$s = (q_1 + q_2)^2, \quad t = (q_1 + q_4)^2, \quad u = (q_1 + q_3)^2,$$

with relation  $s + t + u = -q_1^2 - q_2^2 - q_3^2 - q_4^2 = -16/\kappa^2$ .

This is the Virasoro-Shapiro amplitude for closed strings. Corresponding amplitude for open strings

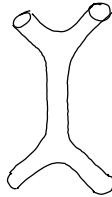
$$A_4 \sim g_s \frac{\Gamma(-1 - \kappa^2 s) \Gamma(-1 - \kappa^2 t)}{\Gamma(+2 + \kappa^2 u)}$$



was proposed (not calculated) earlier by Veneziano. Considered birth of string theory (dual resonance model).

Amplitudes have many desirable features:

- Poles at  $s, t, u = (N - 1)4/\kappa^2$  or  $s, t = (N - 1)/\kappa^2$ , virtual particles with string mass exchanged.



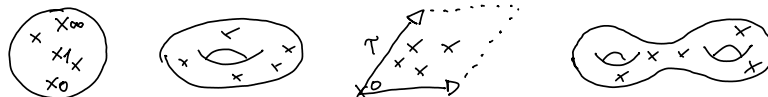
- Residues indicate spin  $J = 2N$  or  $J = N$ . Regge trajectory!
- Soft behaviour at  $s \rightarrow \infty$ . Even for gravitons!
- Manifest crossing symmetry  $s \leftrightarrow t \leftrightarrow u$  or  $s \leftrightarrow t$ . Amazing!

Not possible for QFT with finitely many particles.

### 8.3 String Loops

Result exact as far as  $\alpha'$  is concerned. Free theory in  $\alpha'$ !

However, worldsheet topology matters. String loop corrections for adding handles: higher genus. Power of  $g_s$  reflects Euler characteristic of worldsheet.



**Tree Level.** Worldsheet is sphere or disk with  $n$  punctures. Euler characteristic  $-2 + n$  or  $-1 + n/2$ . 6 global conformal symmetries, integration over  $n - 3$  points.

**One Loop.** Worldsheet is torus with  $n$  punctures. Euler characteristic  $n$ . 2 moduli: integration over Teichmüller space. 2 shifts; integration over  $n - 1$  points.  $2n$  integrations; result: elliptic & modular functions; feasible!

**Two Loops.** Worldsheet is 2-torus with  $n$  punctures. Euler characteristic  $2 + n$ . 6 moduli, no shifts:  $2(n + 3)$  integrations. Hard, but can be done. No higher-loop results available.

## 9 String Backgrounds

Have seen that string spectrum contains graviton. Graviton interacts according to laws of General Relativity. General Relativity is a theory of spacetime geometry. Strings can move in curved backgrounds.

How are strings and gravity related?

- Should we quantise the string background?
- Is the string graviton the same as the Einstein graviton?
- Is there a backreaction between strings and gravity?

### 9.1 Graviton Vertex Operator

Compare graviton as string excitation and background. Assume momentum  $q$  and polarisation  $\epsilon_{\mu\nu}$ .

**Vertex Operator Construction.** Graviton represented by closed string state

$$|\epsilon; q\rangle = \epsilon_{\mu\nu} (\alpha_{-1}^{L,\mu} \alpha_{-1}^{R,\nu} + \alpha_{-1}^{L,\nu} \alpha_{-1}^{R,\mu}) |0; q\rangle.$$

Corresponding vertex operator reads

$$\begin{aligned} \mathcal{O}^{\mu\nu} &= :(\partial X^\mu \bar{\partial} X^\nu + \partial X^\nu \bar{\partial} X^\mu) e^{iq \cdot X}: \\ &\sim : \sqrt{\det -g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu e^{iq \cdot X} :. \end{aligned}$$

Insertion into string worldsheet

$$V = \int d^2\xi \frac{1}{2} \epsilon_{\mu\nu} \mathcal{O}^{\mu\nu}.$$

**Background Metric Construction.** Flat background with plane wave perturbation

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon_{\mu\nu} e^{iq \cdot x} + \dots$$

Strings couple to background by replacement  $\eta_{\mu\nu} \rightarrow G_{\mu\nu}$

$$S = -\frac{1}{2\pi\alpha'^2} \int d^2\xi \sqrt{-\det g} g^{\alpha\beta} \frac{1}{2} G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu.$$

Same replacement  $\eta_{\mu\nu} \rightarrow G_{\mu\nu}$  in Nambu-Goto action.

Perturbation of metric same as vertex operator

$$S = S_0 - \frac{1}{2\pi\alpha'^2} V + \dots$$

**Conclusion.** Graviton mode of string is the same as wave on background.

Quantum string on flat space contains gravitons. Gravitons introduce curvature and deform flat background. String theory contains quantum gravity. Large deformations away from flat background represented by coherent states of gravitons.

String theory can be formulated on any background. String quantisation probes nearby backgrounds. Low-energy physics depends on classical background. Full quantum string theory is background independent, contains all backgrounds as different states (same as QG).

## 9.2 Curved Backgrounds

Consider strings on a curved background  $G_{\mu\nu}(x)$ , curious insight awaits. Action in conformal gauge

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi \frac{1}{2} G_{\mu\nu}(X) \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu.$$

For generic metric  $G$ , e.o.m. for  $X$  are non-linear.

Type of model called non-linear sigma model. String background called target space. Metric field  $G_{\mu\nu}(x)$  is sigma model coupling. Infinitely many couplings (Taylor expansion of  $G$ ).


In most QFT's couplings are renormalised. Problem here:

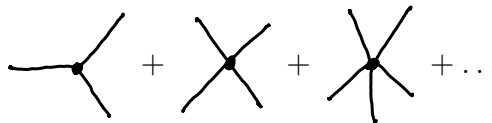
- Classical action has conformal symmetry.
- Conformal symmetry indispensable to remove one d.o.f..
- Renormalised coupling  $G(x, \mu)$  depends on scale  $\mu$ .
- New scale breaks quantum conformal invariance. Anomaly!

**Renormalisation.** Compute the conformal anomaly. Background field quantisation:

- Pick (simple) classical solution  $X_0$  of string e.o.m..
- add perturbations  $X = X_0 + \kappa Y$ . Quantum field  $Y$ .

Expansion of action  $S[X] = S[X_0] + Y^2 + \kappa Y^3 + \dots$  in orders of  $Y$

- Value of classical action  $S[X_0]$  at  $Y^0$  irrelevant.
- No linear term in  $Y$  due to e.o.m. for  $X_0$ .
- Order  $Y^2$  is kinetic term for quantum field  $Y$ . 
- Order  $Y^3, Y^4, \dots$  are cubic, quartic, ... interactions



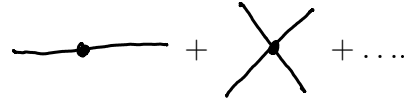
Use target space diffeomorphisms s.t. locally

$$S = - \int \frac{d^2\xi}{2\pi} \eta^{\alpha\beta} (\eta_{\mu\nu} \partial_\alpha Y^\mu \partial_\beta Y^\nu + \frac{1}{3} \kappa^2 R_{\mu\rho\nu\sigma} \partial_\alpha Y^\mu \partial_\beta Y^\nu Y^\rho Y^\sigma)$$



$R_{\mu\rho\nu\sigma}(x)$  is target space curvature tensor.

Kinetic term and quartic vertex:



At one loop we get tadpole diagram. Insert two-point correlator

$$\kappa^2 R_{\mu\rho\nu\sigma} \partial_\alpha Y^\mu(\xi) \partial_\beta Y^\nu(\xi) \langle Y^\rho(\xi) Y^\sigma(\xi) \rangle$$

but we know for  $\xi_1 \rightarrow \xi_2$

$$\langle Y^\rho(\xi_1) Y^\sigma(\xi_2) \rangle \simeq -\eta^{\rho\sigma} \log |\xi_1 - \xi_2|.$$

Not exact, but UV behaviour fixed by conformal symmetry. Logarithmic singularity responsible for renormalisation.  $G_{\mu\nu}$  is running coupling, beta function

$$\frac{\mu \partial G}{\partial \mu} = \beta_{\mu\nu} = \kappa^2 R_{\mu\nu}, \quad R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}.$$

**Anomaly.** Scale dependence breaks conformal symmetry: Trace of stress energy tensor after renormalisation

$$\eta^{\alpha\beta} T_{\alpha\beta} = -\frac{1}{2\kappa^2} \beta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu.$$

Anomaly of Weyl symmetry! (gauge fixed already)

Conformal/Weyl symmetry is essential for correct d.o.f.. Remove by setting  $\beta_{\mu\nu} = 0$ . Einstein equation!

$$R_{\mu\nu} = 0.$$

Quantum strings can propagate only on Einstein backgrounds. General relativity! Spin-2 particles at level 1 are gravitons.

**Higher Corrections.** There are corrections to the beta function from higher perturbative orders in  $\kappa^2$

$$\kappa^2 \left( \text{tadpole diagram} \right) + \kappa^4 \left( \text{circle with dot} + \text{figure-eight} + \text{figure-eight} \right) + \dots$$

$$\beta_{\mu\nu} = \kappa^2 R_{\mu\nu} + \frac{1}{2} \kappa^4 R_{\mu\rho\sigma\kappa} R_\nu{}^{\rho\sigma\kappa} + \dots$$

Also corrections from the expansion in the string coupling  $g_s$ .

Corrections to Einstein equations at Planck scale:  $\beta_{\mu\nu} = 0$ .

### 9.3 Form Field and Dilaton

What about the other (massless) fields? Two-form  $B_{\mu\nu}$  and dilaton scalar  $\Phi$ ?

Two-form couples via antisymmetric combination

$$\frac{1}{2\pi\kappa^2} \int d^2\xi \frac{1}{2} B_{\mu\nu}(X) \varepsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu = \frac{1}{2\pi\kappa^2} \int B.$$

In fact, canonical coupling of two-form to 2D worldsheet. Analogy to charged particle in electromagnetic field. String has two-form charge.

Dilaton couples to worldsheet Riemann scalar

$$\frac{1}{4\pi} \int d^2\xi \sqrt{-\det g} \Phi(X) R[g].$$

Interesting for several reasons:

- Euler characteristic  $\chi$  of the worldsheet appears.
- Not Weyl invariant.
- Scalar can mix with gravity.
- Can get away from 26 dimensions.

**Low-Energy Effective Action.** First discuss the various beta functions (trace of renormalised stress energy tensor  $T$ )

$$g^{\alpha\beta} T_{\alpha\beta} = -\frac{1}{2\kappa^2} (\sqrt{-\det g} \beta_{\mu\nu}^G \eta^{\alpha\beta} + \beta_{\mu\nu}^B \varepsilon^{\alpha\beta}) \partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2} \beta^\Phi R[g]$$

with

$$\begin{aligned} \beta_{\mu\nu}^G &= \kappa^2 R_{\mu\nu} + 2\kappa^2 D_\mu D_\nu \Phi - \frac{1}{4} \kappa^2 H_{\mu\rho\sigma} H_\nu^{\rho\sigma}, \\ \beta_{\mu\nu}^B &= -\frac{1}{2} \kappa^2 D^\lambda H_{\mu\nu\lambda} + \kappa^2 D^\lambda \Phi H_{\mu\nu\lambda}, \\ \beta^\Phi &= -\frac{1}{2} \kappa^2 D^2 \Phi + \kappa^2 D^\mu \Phi D_\mu \Phi - \frac{1}{24} \kappa^2 H_{\mu\nu\rho} H^{\mu\nu\rho}. \end{aligned}$$

Quantum string consistency requires  $\beta^G = \beta^B = \beta^\Phi = 0$ . Standard equations for graviton, two-field and scalar. Follow from an action

$$S \sim \int d^{26}x \sqrt{-\det g} e^{-2\Phi} \left( R - \frac{1}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial^\mu \Phi \partial_\mu \Phi \right).$$

String low-energy effective action. Encodes low-energy physics of string theory. Further corrections from curvature and loops.

Trivial solution:  $G = \eta$ ,  $B = 0$ ,  $\Phi = \Phi_0$  (flat background). Can also use torus compactification to reduce dimensions.

**String Coupling.** Suppose  $\Phi = \Phi_0$  is constant, then dilaton coupling term is topological

$$\int d^2\xi \sqrt{-\det g} R[g] \sim \chi.$$

Measures Euler characteristic  $\chi = 2h - 2$  of world sheet.

Set  $g_s = e^{i\Phi_0}$ . Then action yields  $\chi$  factors of  $g_s$ .

$$e^{iS} \simeq e^{i\Phi_0\chi} = g_s^\chi.$$

Expansion in worldsheet topology.

$$g_s^{-2} \text{ (sphere) } + g_s^0 \text{ (torus) } + g_s^2 \text{ (genus 2 surface)}$$

String coupling  $g_s$  determined through background: Asymptotic value  $\Phi_0$  of dilaton field  $\Phi$ .

**String Frame.** Notice unusual factor of  $\exp(-2\Phi)$  in  $S$ .

Scalar degrees of freedom can mix with metric. Could as well define

$$G'_{\mu\nu} = f(\Phi)G_{\mu\nu}.$$

Remove  $\exp(-2\Phi)$  through suitable choice of  $f$ . Go from “string frame” to “Einstein frame”. Standard kinetic terms for all fields.

**Noncritical Strings.** We have seen earlier that  $D \neq 26$  breaks Weyl symmetry.  $D$  enters in effective action as worldsheet cosmological constant

$$S = \dots \left( R - \frac{1}{2} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial^\mu\Phi\partial_\mu\Phi - \frac{2}{3}\kappa^{-2}(D - 26) \right).$$

Can have  $D < 26$ , but requires Planck scale curvature.

**Dilaton Scaling.** Dilaton coupling to worldsheet is not Weyl invariant and has unconventional power of  $\kappa$ .

- Consistent choice.
- Moves classical Weyl breakdown to one loop. Cancel quantum anomalies of other fields.

## 9.4 Open Strings

Open strings lead to additional states, fields and couplings.

- Additional string states; e.g. massless vectors (photon):

$$|\zeta; q\rangle = \zeta_\mu \alpha_{-1}^\mu |0; q\rangle.$$

- Additional vertex operators; e.g. photon

$$V[\zeta, q] \sim \int d\tau \zeta_\mu \partial_\tau X^\mu \exp(iq \cdot X)$$

- Additional fields to couple to string ends.

Background couplings can be identified as for closed strings. Vertex operator has same effect as background field.

Coupling depends on string boundary conditions:  $Dp$ -brane.

**Neumann Boundaries.** For all coordinates  $X^a$ ,  $a = 0, \dots, p$ , with Neumann conditions: couple a one-form gauge field  $A$  to end of string

$$\int_{\text{end}} d\tau \dot{X}^a A_a(X) = \int_{\text{end}} A.$$

- natural coupling of a charged point-particle to gauge field.
- string end is a charged point-like object.

Gauge field  $A_a$  exists only on  $Dp$ -brane. Okay since string ends constrained to  $Dp$  brane.

Classical coupling of  $A$  respects Weyl symmetry. Quantum anomaly described by beta function

$$\beta_a^A \sim \kappa^4 \partial^b F_{ab}$$

Absence of conformal anomaly requires Maxwell  $\partial^b F_{ab} = 0$ . Associated low-energy effective action

$$S \sim -\kappa^4 \int d^{p+1}x \frac{1}{4} F_{ab} F^{ab}.$$

For planar  $Dp$ -brane can also include higher corrections in  $\kappa$ . Born–Infeld action:

$$S \sim \int d^{p+1}x \sqrt{-\det(\eta_{ab} + 2\pi\kappa^2 F_{ab})}.$$

Leading order is Maxwell kinetic term. Corrections at higher orders in  $\kappa$ .

**Dirichlet Boundaries.** Coupling of Dirichlet directions  $X^m$ ,  $m = p + 1, \dots, D - 1$ , different.

- $X^m$  fixed, but  $X'^m$  can be used.
- Dual field  $Y_m$  describes transverse  $Dp$ -brane displacement.
- $Dp$ -branes are dynamical objects!

Beta function at leading order: massless scalar

$$\beta_a \sim \partial^m \partial_m Y_a$$

Effective action for higher orders: Dirac–Born–Infeld action

$$S \sim \int d^{p+1}x \sqrt{-\det(g_{ab} + 2\pi\kappa^2 F_{ab})}.$$

Induced WS metric  $g_{ab} = \partial_a Y^\mu \partial_b Y_\mu$ . Embedding coordinates  $Y$  for  $Dp$ -brane.  
Combination of

- Dirac action for  $p$ -branes and
- Born–Infeld action for gauge fields.

**D-Branes in a Curved Background.** Can even add effect of close string fields.

$$S \sim \int d^{p+1}x e^{-\Phi} \sqrt{-\det(g_{ab} + 2\pi\kappa^2 F_{ab} + B_{ab})}.$$

- $g_{ab}$  is induced metric from curved background.
- $B_{ab}$  is pull back of 2-form field  $B_{\mu\nu}$  to  $Dp$ -brane.
- combination  $2\pi\alpha' F_{ab} + B_{ab}$  is gauge invariant.
- dilaton couples as prefactor like for closed string.

**Coincident Branes.** For  $N$  coincident branes gauge group enlarges from  $U(1)^N$  to  $U(N)$ .

Gauge field should couple via Wilson line

$$\text{T exp} \int_{\text{end}} A.$$

Resulting effective action at leading order is

$$S \sim \int d^{p+1}x \text{tr} \left( -\frac{1}{4}(F_{ab})^2 + \frac{1}{2}(D_a Y_m)^2 + \frac{1}{4}[Y_m, Y_n]^2 \right).$$

Yang–Mills, massless adjoint scalars, quartic interactions.

## 9.5 Two-Form Field of a String

We have seen that strings couple to various fields. A string also generates a field configuration. Analogy: charged point particle generates Coulomb potential.

**Fundamental String.** Consider an infinite straight string along 0, 1 directions: 1-brane. Generates a two-form potential

$$B = (f^{-1} - 1)dx^0 \wedge dx^1$$

Interactions with metric  $G$  and dilaton  $\Phi$  require

$$ds^2 = f^{-1}ds_2^2 + ds_{D-2}^2, \quad e^{2\Phi} = f^{-1}.$$

The function  $f$  with  $r^2 = x_2^2 + \dots + x_{D-1}^2$  reads

$$f = 1 + \frac{g_s^2 N \kappa^{D-4}}{r^{D-4}}.$$

This satisfies the low-energy effective string e.o.m. because  $f$  is a harmonic function.

Note: Source at the location of the string ( $r = 0$ ).

- E.o.m. follow from combination of spacetime action and worldsheet coupling to two-form

$$\int_D H \wedge *H + \int_2 B$$

Source term  $\delta^{D-2}(r)$  absorbed by worldsheet.

- Charge of string measured by Gauss law via  $*H$ . Put  $(D - 3)$ -dimensional sphere at fixed  $r$ .

$$Q = \int_{D-3} *H = N.$$

Above string has  $N$  units of charge (quantised).

The fundamental string is not a D1-brane: Open strings do not end on it. It is the string itself.

Solutions with more than one centre permissible.

**Magnetic Brane.** Another solution of the string effective e.o.m. describes a  $(D - 5)$ -brane. It uses a dual  $(D - 4)$ -form potential  $C$  defined through

$$H = dB, \quad *H = dC.$$

It carries magnetic charge

$$Q = \int_3 H.$$

The source is located on the  $(D - 5)$ -brane(s). The coupling of  $(D - 5)$ -branes to  $C$  compensates source.

## 10 Superstrings

Until now, encountered only bosonic d.o.f. in string theory. Matter in nature is dominantly fermionic. Need to add fermions to string theory.

Several interesting consequences:

- Supersymmetry inevitable.
- Critical dimension reduced from  $D = 26$  to  $D = 10$ .
- Increased stability.
- Closed string tachyon absent. Stable D-branes.
- Several formulations related by dualities.

### 10.1 Supersymmetry

String theory always includes spin-2 gravitons. Fermions will likely include spin- $\frac{3}{2}$  gravitini  $\rightarrow$  supergravity. Spacetime symmetries extended to supersymmetry.

**Super-Poincaré Algebra.** Super-Poincaré algebra is an extension of Poincaré algebra.

Poincaré: Lorentz rotations  $M_{\mu\nu}$ , translations  $P_\mu$ .

$$[M, M] \sim M, \quad [M, P] \sim P, \quad [P, P] = 0.$$

Super-Poincaré: Odd super-translation  $Q_m^I$  ( $a$ : spinor)

$$[M, Q] \sim Q, \quad [Q, P] = 0, \quad \{Q_m^I, Q_n^J\} \sim \delta^{IJ} \gamma_{mn}^\mu P_\mu.$$

$\mathcal{N}$ : rank of supersymmetry  $I = 1, \dots, \mathcal{N}$ .

$Q$  relates particles of

- of different spin,
- of different statistics,

and attributes similar properties to them. Symmetry between “forces” and “matter”.

More supersymmetry, higher spin particles.

- gauge theory (spin  $\leq 1$ ):  $\leq 16$   $Q$ 's.
- gravity theory (spin  $\leq 2$ ):  $\leq 32$   $Q$ 's.

**Superspace.** Supersymmetry is symmetry of superspace. Add anticommuting coordinates to spacetime  $x^\mu \rightarrow (x^\mu, \theta_I^a)$ . Superfields: expansion in  $\theta$  yields various fields

$$F(x, \theta) = F_0(x) + \theta_I^m F_m^I(X) + \theta^2 \dots + \dots + \theta^{\dim \theta}.$$

Package supermultiplet of particles in a single field.

**Spinors.** Representations of  $Spin(D-1, 1)$  (Clifford).

Complex spinors (Dirac) in  $(3+1)D$  belong to  $\mathbb{C}^4$ . Can split into chiral spinors (Weyl):  $\mathbb{C}^2 \oplus \mathbb{C}^2$ . Reality condition (Majorana):  $\text{Re}(\mathbb{C}^2 \oplus \bar{\mathbb{C}}^2) = \mathbb{C}^2$ .

Spinors in higher dimensions:

- spinor dimension times 2 for  $D \rightarrow D+2$ .
- chiral spinors (Weyl) for  $D$  even.
- real spinors (Majorana) for  $D = 0, 1, 2, 3, 4 \pmod{8}$ .
- real chiral spinors (Majorana–Weyl) for  $D = 2 \pmod{8}$ .

Maximum dimensions:

- $D = 10$ : real chiral spinor with 16 components (gauge).
- $D = 11$ : real spinor with 32 components (gravity bound).

**Super–Yang–Mills Theory.**  $\mathcal{N} = 1$  supersymmetry in  $D = 10$  Minkowski space:

- gauge field  $A_\mu$ : 8 on-shell d.o.f..
- adjoint real chiral spinor  $\Psi_m$ : 8 on-shell d.o.f..

Simple action

$$S \sim \int d^{10}x \text{tr} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \gamma_{mn}^{\mu} \Psi^n D_\mu \Psi^n \right).$$

**Supergravity Theories.** Four relevant models:

- $\mathcal{N} = 1$  supergravity in 11D: M-Theory.
- $\mathcal{N} = (1, 1)$  supergravity in 10D: Type IIA supergravity.
- $\mathcal{N} = (2, 0)$  supergravity in 10D: Type IIB supergravity.
- $\mathcal{N} = (1, 0)$  supergravity in 10D: Type I supergravity.

Fields always 128+128 d.o.f. (type I: half, SYM only 8+8):

type	gr.	[4]	[3]	[2]	[1]	sc.	gravitini	spinors
M	1	0	1	0	0	0	1	0
IIA	1	0	1	1	1	1	(1,1)	(1,1)
IIB	1	1	0	2	0	2	(2,0)	(2,0)
I	1	0	0	1	0	1	(1,0)	(1,0)
SYM	0	0	0	0	1	0	0	(0,1)

M-theory has no 2-form and no dilaton: no string theory. Type IIA, IIB and I have 2-form and dilaton: strings?!

## 10.2 Green–Schwarz Superstring

Type II string: Add fermions  $\Theta_I^m$  to worldsheet. Equal/opposite chirality: IIB/IIA



**Action.** Supermomentum  $\Pi_\alpha^\mu = \partial_\alpha X^\mu + \delta^{IJ} \gamma_{mn}^\mu \Theta_I^m \partial_\alpha \Theta_J^n$ .

$$S \sim \int d^2\xi \sqrt{-\det g} g^{\alpha\beta} \eta_{\mu\nu} \Pi_\alpha^\mu \Pi_\beta^\nu + \int \left( (\Theta^1 \gamma_\mu d\Theta^1 - \Theta^2 \gamma_\mu d\Theta^2) dX^\mu + \Theta^1 \gamma_\mu d\Theta^1 \Theta^2 \gamma^\mu d\Theta^2 \right).$$

Action has kappa symmetry (local WS supersymmetry). Only in  $D = 10$ !

Note: fermions  $\Theta$  have first and second class constraints. Non-linear equations of motion. In general difficult to quantise canonically. Conformal gauge does not resolve difficulties.

**Light-Cone Gauge.** Convenient to apply light-cone gauge. Simplifies drastically: quadratic action, linear e.o.m.

$$S \sim \int d^2\xi \left( \partial_L \vec{X} \cdot \partial_R \vec{X} + \frac{1}{2} \Theta_1 \cdot \partial_R \Theta_1 + \frac{1}{2} \Theta_2 \cdot \partial_L \Theta_2 \right)$$

Bosons  $\vec{X}$  with  $\partial_L \partial_R \vec{X} = 0$

- Vector of transverse  $SO(8)$ :  $\mathbf{8}_v$
- Left and right moving d.o.f.

Fermions  $\Theta_1, \Theta_2$  with  $\partial_R \Theta_1 = 0$  and  $\partial_L \Theta_2 = 0$

- Real chiral spinor of transverse  $SO(8)$ :  $\mathbf{8}_s$  or  $\mathbf{8}_c$ . Equal/opposite chiralities for IIB/IIA:  $\mathbf{8}_s + \mathbf{8}_s$  or  $\mathbf{8}_s + \mathbf{8}_c$
- Left and right moving d.o.f. in  $\Theta_1$  and  $\Theta_2$ , respectively.

**Spectrum.** Vacuum energy and central charge:

- 8 bosons and 8 fermions for L/R:  $a_{L/R} = 8\zeta(1) - 8\zeta(1) = 0$ . no shift  $a$  for  $L_0$  constraint. Level zero is massless! No tachyon!
- $c = 10 + 32 \frac{1}{2} = 26$  (fermions count as  $\frac{1}{2}$  due to kappa).
- Super-Poincaré anomaly cancels.

Expansion into bosonic modes  $\alpha_n$  and fermionic modes  $\beta_n$ .  $n < 0$ : creation,  $n = 0$ : zero mode,  $n > 0$ : annihilation.

Zero modes and vacuum:

- $\alpha_0$  is c.o.m. momentum:  $\vec{q}$ .
- $\beta_0$  transforms the vacuum state:

$$\begin{aligned} \beta \text{ chiral } (\mathbf{8}_s) : & \quad \mathbf{8}_v \leftrightarrow \mathbf{8}_c \quad \text{vacuum} \rightarrow |\mathbf{8}_v + \mathbf{8}_c, q\rangle \\ \beta \text{ anti-chiral } (\mathbf{8}_c) : & \quad \mathbf{8}_v \leftrightarrow \mathbf{8}_s \quad \text{vacuum} \rightarrow |\mathbf{8}_v + \mathbf{8}_s, q\rangle \end{aligned}$$

Spectrum at level zero: massless

- Type IIA closed:  $(\mathbf{8}_v + \mathbf{8}_s) \times (\mathbf{8}_v + \mathbf{8}_c)$  (IIA supergravity)

$$\begin{aligned} \mathbf{8}_v \times \mathbf{8}_v + \mathbf{8}_s \times \mathbf{8}_c &= (\mathbf{35}_v + \mathbf{28}_v + \mathbf{1}) + (\mathbf{56}_v + \mathbf{8}_v), \\ \mathbf{8}_v \times \mathbf{8}_s + \mathbf{8}_v \times \mathbf{8}_c &= (\mathbf{56}_s + \mathbf{8}_c) + (\mathbf{56}_c + \mathbf{8}_s). \end{aligned}$$

- Type IIB closed:  $(\mathbf{8}_v + \mathbf{8}_c) \times (\mathbf{8}_v + \mathbf{8}_c)$  (IIB supergravity)

$$\begin{aligned}\mathbf{8}_v \times \mathbf{8}_v + \mathbf{8}_c \times \mathbf{8}_c &= (\mathbf{35}_v + \mathbf{28}_v + \mathbf{1}) + (\mathbf{35}_c + \mathbf{28}_v + \mathbf{1}), \\ \mathbf{8}_v \times \mathbf{8}_s + \mathbf{8}_v \times \mathbf{8}_s &= (\mathbf{56}_s + \mathbf{8}_c) + (\mathbf{56}_s + \mathbf{8}_c).\end{aligned}$$

- Type I closed:  $(\mathbf{8}_v + \mathbf{8}_c) \times (\mathbf{8}_v + \mathbf{8}_c) \bmod \mathbb{Z}_2$  (I supergravity)

$$(\mathbf{35}_v + \mathbf{28}_v + \mathbf{1}) + (\mathbf{56}_s + \mathbf{8}_c).$$

- Type I open:  $\mathbf{8}_v + \mathbf{8}_c$  (SYM).

### 10.3 Ramond–Neveu–Schwarz Superstring

There is an alternative formulation for the superstring: RNS. Manifest worldsheet rather than spacetime supersymmetry!

**Action.** Action in conformal gauge:

$$S \sim \int d^2\xi \eta_{\mu\nu} \left( \frac{1}{2} \partial_L X^\mu \partial_R X^\nu + i \Psi_L^\mu \partial_R \Psi_L^\nu + i \Psi_R^\mu \partial_L \Psi_R^\nu \right).$$

- action is supersymmetric.
- fermions are worldsheet spinors but spacetime vectors.

Bosons as before. Fermions can be periodic or anti-periodic.

**Ramond Sector.**  $\Psi(\sigma + 2\pi) = \Psi(\sigma)$  periodic.

- Fermion modes  $\beta_n$  as for bosons.
- Vacuum is a real 32-component fermionic spinor.
- $a = -\frac{1}{2}8\zeta(1) + \frac{1}{2}8\zeta(1) = 0$ .
- GSO projection: only chiral/anti-chiral states are physical!

**Neveu–Schwarz Sector.**  $\Psi(\sigma + 2\pi) = -\Psi(\sigma)$  anti-periodic.

- Half-integer modes for fermions:  $\beta_{n+1/2}$ .
- Vacuum is a bosonic scalar.
- $a = -\frac{1}{2}8\zeta(1) - \frac{1}{4}8\zeta(1) = \frac{1}{2}$ .
- GSO projection: physical states require  $\beta^{2n+1}$ . No tachyon!

**String Models.** IIB/IIA strings for equal/opposite chiralities in L/R sectors.

Independent choice for left/right-movers in closed string. Four sectors: NS-NS, RN-S, NS-R, R-R. Independent vacua.

**Superconformal Algebra.** (Left) stress-energy tensor and conformal supercurrent:

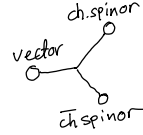
$$T_L = \partial_L X \cdot \partial_L X + \frac{i}{2} \Psi_L \cdot \partial_L \Psi_L, \quad J_L = \Psi_L \cdot \partial_L X$$

Superconformal algebra  $L_n, G_r$  ( $2r$  is even/odd for R/NS):

$$\begin{aligned} [L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{8}cm(m^2 - 1)\delta_{m+n}, \\ [L_m, G_r] &= (\frac{1}{2}m - r)G_{m+r}, \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{1}{2}c(r^2 - \frac{1}{4})\delta_{r+s}. \end{aligned}$$

$c = D$  (conventional factor  $\frac{3}{2}$  in  $c$  for super-Virasoro).

**Comparison.** GS and RNS approach yield the same results. In light cone gauge: related by  $SO(8)$  triality



Compare features of both approaches:

	GS	RNS
fermions are spinors in worldsheet supersymmetry	target space (✓)	worldsheet manifest
superconformal field theory	×	✓
target space supersymmetry	manifest	(✓)
supergravity couplings	all	some (NS-NS)
spacetime covariant	×	(✓)

Third approach exists: Pure spinors (Berkovits). Introduce auxiliary bosonic spinor  $\lambda$  satisfying  $\lambda\gamma^\mu\lambda = 0$ . Shares benefits of GS/RNS; covariant formulation.

## 10.4 Branes

Open superstrings couple to D-branes. Open string spectrum carries D-brane fluctuations.

- massless:  $\mathcal{N} = 1$  Super-Yang-Mills reduced to  $(d + 1)D$ .
- heavy string modes.
- sometimes: scalar tachyon.

**Stable Dp-Branes.** D-branes can be stable or decay. Open string tachyon indicates D-brane instability.

- D-branes in bosonic string theory are unstable.
- Dp-branes for IIB superstring are stable for  $p$  odd.
- Dp-branes for IIA superstring are stable for  $p$  even.

- T-duality maps between IIA and IIB.

Stability is related to supersymmetry. Boundary conditions break symmetry

- Lorentz:  $SO(9, 1) \rightarrow SO(d, 1) \times SO(9 - d)$ .
- 16 supersymmetries preserved for  $p$  odd/even in IIB/IIA.
- no supersymmetries preserved for  $p$  even/odd in IIB/IIA.

Supersymmetry removes tachyon; stabilises strings.

**Supergravity  $p$ -Branes.** D-branes are non-perturbative objects. Not seen perturbatively due to large mass.

Stable  $Dp$ -branes have low-energy limit as supergravity solutions.

$p$ -brane supported by  $(p + 1)$ -form, gravity and dilaton.

- IIB/IIA have dilaton and two-form (NS-NS sector).
- IIB/IIA has forms of even/odd degree (R-R sector); relevant for stable  $Dp$ -branes.

Features:

- $p$ -branes carry  $(p + 1)$ -form charge. charge prevents  $p$ -branes from evaporating.
- charge density equals mass density.
- 16/32 supersymmetries preserved. 1/2 BPS condition.
- Non-renormalisation theorem for 1/2 BPS:  $p$ -branes same at weak/intermediate/strong coupling. BPS  $p$ -branes describe  $Dp$ -branes exactly.

**Type-I Superstring.** Consider open strings on D9-branes.

Gravity and gauge anomaly cancellation requires:

- gauge group of dimension 496.
- some special charge lattice property.

Two solutions:  $SO(32)$  and  $E_8 \times E_8$ . Here:  $SO(32)$ . Breaks 1/2 supersymmetry: Type I.

- Sometimes considered independent type of superstring.
- Or: IIB, 16 D9 branes, space-filling orientifold-plane.

## 10.5 Heterotic Superstring

Two further superstring theories.

Almost no interaction between left and right movers. Exploit:

- left-movers as for superstring: 10D plus fermions.
- right-movers as for bosonic string: 26D (16 extra).

Heterotic string. 16 supersymmetries.

Anomaly cancellation requires gauge symmetry:

- HET-O:  $SO(32)$  or

- HET-E:  $E_8 \times E_8$ .

Gauge group supported by 16 internal d.o.f..

HET-E interesting because  $E_8$  contains potential GUT groups:

$$E_5 = SO(10), \quad E_4 = SU(5), \quad E_3 = SU(3) \times SU(2).$$

## 10.6 Dualities

Dualities relate seemingly different superstring theories.

- T-duality: time vs. space duality on worldsheet.
- S-duality: analog of electro-magnetic duality.

Dualities considered exact because of supersymmetry. Tests.

**A Unique Theory.** Dualities related various superstrings:

- T-duality: IIA  $\leftrightarrow$  IIB; HET-E  $\leftrightarrow$  HET-O
- S-duality: HET-O  $\leftrightarrow$  Type I; IIB  $\leftrightarrow$  IIB

Furthermore IIA and HET-E at strong coupling: 11D supergravity theory (with membrane).

Suspect underlying 11D theory called “M-theory”. Superstring theories as various limits of M-theory.

**Mirror Symmetry.** Dualities applied to curved string backgrounds: Curved spacetimes with

- inequivalent metrics can have
- equivalent string physics.

E.g.: T-duality between large and small circles. Many examples for Calabi–Yau manifolds.

**String/Gauge Duality.** Some low-energy effective theories can become exact.

String physics at the location of a brane described exactly by corresponding YM theory.

Example:  $N$  coincident D3-branes in IIB string theory. Effective theory:  $\mathcal{N} = 4$  Super-Yang–Mills theory in 4D.

## 11 AdS/CFT Correspondence

Conjectured exact duality between string theory and CFT.

- Remarkable!
- Precise formulation of a string/gauge duality.
- Holographic. Different number of spacetime dimensions.
- Main example:  $AdS_5 \times S^5$  string and  $\mathcal{N} = 4$  SYM.

### 11.1 Stack of D3-Branes

Consider 3-brane solution of IIB supergravity ( $4 x \parallel, 6 y \perp$ )

$$ds^2 = h^{-1/2} dx^2 + h^{1/2} dy^2, \quad H_5 = h^{-2} dh dx^4 + h^{-2} *(dh dx^4),$$

with harmonic function  $h(y) = 1 + \alpha N/|y|^4$ .

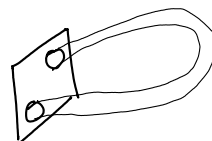
IIB string theory background with stack of  $N$  D3-branes. Low-energy brane physics described by  $U(N)$   $\mathcal{N} = 4$  SYM.

Now approach brane at  $y = 0$ . Alternatively send  $N \rightarrow \infty$ .

- Harmonic function limits to  $h(y) = \alpha N/|y|^4$ .
- Background becomes  $AdS_5 \times S^5$  with 5-form flux.
- $S^5$  at constant  $|y|$ .  $AdS_5$  combined from  $x$  and  $|y|$ .

Claims: AdS/CFT correspondence (Maldacena)

- 3-brane at boundary of  $AdS_5$  space.
- Non-brane modes decouple.
- Boundary physics described exactly by  $U(N)$   $\mathcal{N} = 4$  SYM.
- Open string on boundary can probe bulk  $AdS_5 \times S^5$  strings.



- Precise matching of all observables in both models.
- Map of coupling constants  $(\kappa/R, g_s)$  with  $(g_{YM}, N)$ .

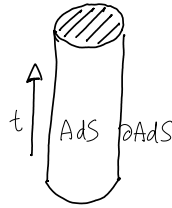
### 11.2 Anti-de Sitter Geometry

Anti-de Sitter space  $AdS_d$  is curved spacetime:

- Constant scalar curvature.
- Analogous to sphere and hyperbolic space

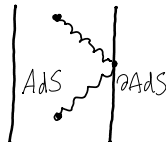
curvature	+	-
Euclidean	S	H
Minkowski	dS	AdS

- Isometry group:  $SO(d-1, 2)$ . Same as conformal group in  $d-1$  dimensions.
- Topology: Solid cylinder  $\mathbb{R} \times D^{d-1}$



Boundary: Cylinder surface  $\mathbb{R} \times S^{d-1}$ .

- time-like geodesics never reach boundary.
- space-like geodesics reach boundary at infinite distance.
- light-like geodesics reach boundary in finite time. bulk and boundary interact via massless fields.



### 11.3 $\mathcal{N} = 4$ Super-Yang-Mills

Maximally supersymmetric gauge theory in 4D. Dimensional reduction from  $\mathcal{N} = 1$  SYM in  $D = 10$ . Fields:

- gauge field,
- 4 adjoint Dirac fermions,
- 6 adjoint scalars.

Remarkable properties:

- no running coupling,  $\beta = 0$ .
- exact 4D superconformal symmetry; 4D (S)CFT.
- ...

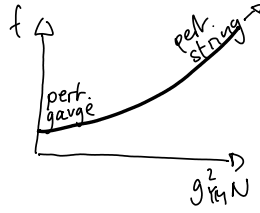
### 11.4 Tests

Want to test AdS/CFT correspondence. Predictions:

- String spectrum matches with spectrum of local operators.
- String and gauge correlation functions match.

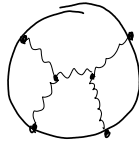
Problem: Strong/weak coupling duality.

- Weakly coupled strings is strongly coupled gauge theory.
- Weakly coupled gauge theory is strongly coupled strings.



Test BPS quantities, protected (independent of coupling).

- Supergravity modes agree with BPS operators.
- Supergravity correlators match with BPS correlators.



What about other quantities?

- String and gauge theory appear integrable at large  $N$ .
- Integrability: Hidden symmetry to constrain dynamics.
- Can compute observables efficiently even at finite coupling.
- Precise agreement found in all performed tests.

Other tests performed, e.g. Wilson loops vs. string area.



## 1. On the importance of quantum gravity (easy)

Let us get some intuition on the order of magnitudes:

- a) Consider a gravitational atom, an electron bound to a neutron by the gravitational force. Electromagnetic dipole effects can be neglected. Perform a semiclassical calculation to determine the radius of the orbit of the electron (first Bohr radius). Relate this radius to an appropriate distance in physics.
- b) In “natural units”, where  $\hbar$ ,  $G$  and  $c$  are set to 1, a stellar black hole radiates like a black body at a temperature given by  $kT = 1/8\pi M$ . Give the temperature in SI units (reinsert  $G$ ,  $\hbar$  and  $c$ ) and calculate the temperature of a black hole weighing one solar mass.

## 2. Relativistic point particle (intermediate)

The action of a relativistic point particle is given by

$$S_{\text{rp}} = -\alpha \int_{\mathcal{P}} ds$$

with the relativistic line element

$$ds^2 = -\eta_{\mu\nu} dX^\mu dX^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

and  $\alpha$  a (yet to be determined) constant. The path  $\mathcal{P}$  between two points  $X_1^\mu$  and  $X_2^\mu$  can be parametrised by a parameter  $\tau$ . The integral over the line element  $ds$  becomes an integral over the parameter

$$S_{\text{rp}} = -\alpha \int_{\tau_1}^{\tau_2} d\tau \sqrt{-\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \tau}}. \quad (*)$$

- a) Parametrise the path by the time coordinate  $t$  and take the non-relativistic limit  $|\dot{\vec{x}}| \ll c$  to determine the value of the constant  $\alpha$ . Characterise the appearing terms.
- b) Derive the equations of motion by varying the action in (\*). (You may set  $c = 1$  from now on.) *Hint:* Calculate the canonically conjugate momentum  $P_\mu$  first.
- c) Show that the form of the action is invariant under reparametrisations  $\tau' = f(\tau)$ . This is what we call *manifestly* invariant.
- d) Consider an electrically charged particle with charge  $q$ . In the presence of an external gauge field  $A_\mu$  there is an additional term in the action governing the interaction between particle and field given by

$$S_{\text{em}} = \frac{q}{c} \int d\tau A_\mu(X) \frac{\partial X^\mu}{\partial \tau}.$$

Find the variation of  $A^\mu(X)$  under a variation of the path  $\delta X^\mu$ . Vary the action  $S = S_{\text{rp}} + S_{\text{em}}$  w.r.t.  $X^\mu$  to find the equations of motion for the particle. *Hint:* Use  $P_\mu$  from above to simplify the expression.

### 3. Polynomial action (intermediate – hard)

There is another way to write the action of a relativistic particle. We introduce an auxiliary field called vierbein (or “einbein” in this case)  $e$  along the worldline of the particle and rewrite the action in the form

$$S_{\text{pp}} = \int d\tau (e^{-1} \dot{X}^2 - m^2 e).$$

- a) Show that  $S_{\text{pp}}$  is equivalent to  $S_{\text{rp}}$  above by eliminating the einbein from the action.
- b) Derive the equations of motion by varying  $S_{\text{pp}}$  with respect to  $X$  and  $e$ .
- c) Show that  $S_{\text{pp}}$  is invariant under infinitesimal reparametrisations  $\delta\tau = -\epsilon(\tau)$  to linear order in  $\epsilon$ . First find the correct transformation of  $X^\mu$ . The einbein transforms like (can you derive it?)

$$\delta e = \partial_\tau(\epsilon(\tau)e).$$

- d) Reparametrisation invariance is a gauge invariance. Thus by fixing a gauge we can eliminate one degree of freedom. Assume a gauge in which  $e$  is constant. Show that  $e$  can be written like

$$e = \frac{\ell}{\tau_2 - \tau_1},$$

where  $\ell$  is the invariant length of the worldline for a path starting at  $X^\mu(\tau_1)$  and ending at  $X^\mu(\tau_2)$ . *Hint:* Meditate on the role of the einbein and on how to define  $\ell$ .

## 1. Symmetries of the classical string (intermediate)

In this exercise we examine the classical symmetries of the Polyakov string

$$S_P = -\frac{T}{2} \int d^2\xi \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.$$

We start with the global symmetries – Lorentz and translational symmetry – and proceed to gauge symmetries – reparametrisation and Weyl symmetry.

a) Consider the transformation

$$X^\mu \rightarrow \Lambda^\mu{}_\nu X^\nu + a^\mu$$

which is a combination of a Lorentz transformation and a translation, a.k.a. a Poincaré transformation. Using the Noether procedure show that in conformal gauge  $g_{\alpha\beta} = \eta_{\alpha\beta}$  the Noether currents corresponding to these symmetries are given by

$$\mathcal{P}_\mu^\alpha = T \partial^\alpha X_\mu, \quad \mathcal{J}_{\mu\nu}^\alpha = \mathcal{P}_\mu^\alpha X_\nu - \mathcal{P}_\nu^\alpha X_\mu.$$

- b) Find and identify the conserved charges associated with Lorentz boosts and time translations. *Hint:* For the Lorentz boost assume  $X^0 = t$ .
- c) Show that the Polyakov string action is invariant under a reparametrisation  $\xi^\alpha \rightarrow \tilde{\xi}^\alpha(\xi)$ .
- d) Show that the Polyakov string is also invariant under Weyl transformations: local length-changing but angle preserving transformations of the metric  $g_{\alpha\beta} \rightarrow e^{2\omega(\xi)} g_{\alpha\beta}$ .
- e) Consider an infinitesimal Weyl transformation

$$\delta g_{\alpha\beta} = 2\omega g_{\alpha\beta} \quad \text{and} \quad \delta X^\mu = 0$$

and show that Weyl symmetry implies the vanishing of the trace of the worldsheet energy-momentum tensor

$$T^\alpha{}_\alpha = 0.$$

*Hint:* The variation of the determinant is given by

$$\delta \det g = -\det g g_{\alpha\beta} \delta g^{\alpha\beta}.$$

→

## 2. Classical spinning strings (easy – intermediate)

The classical solution for the wave equation is given by

$$X^\mu(\sigma, \tau) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma),$$

the constraints by

$$\dot{X} \cdot X' = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0.$$

It will be beneficial to work in static gauge  $X^0(\sigma, \tau) = R\tau$  ( $\tau$  being the worldsheet time).

a) Show that

$$\begin{aligned} X^0 &= R\tau \\ X^1 &= R \cos(\sigma) \cos(\tau) \\ X^2 &= R \cos(\sigma) \sin(\tau) \end{aligned}$$

can be written in the form of the general solution of the wave equation and that it fulfils the constraints. Calculate the energy  $P^0 = E$  and the angular momentum  $J_{ij}$  of the solution.

b) Show that

$$\begin{aligned} X^0 &= R\tau \\ X^1 &= R \cos(\sigma) \cos(\tau) \\ X^2 &= R \cos(2\sigma) \sin(2\tau) \end{aligned}$$

can be written in the form of the general solution of the wave equation but does not fulfil the constraint equations.

c) (optional) Closed strings can develop cusps. These points  $\sigma_0$  on the string are indicated by a singularity in the parametrisation

$$\frac{\partial \vec{X}}{\partial t}(\sigma_0, t) = 0.$$

Show that the string reaches the speed of light at a cusp. Moreover, show that cusps move perpendicular to the direction of the string.

d) (advanced) Explain why cusps form generically in  $3 + 1$  dimensions but not so in higher dimensions.

## 1. Light cone tensors (easy)

We want to derive some relations between Lorentz tensors  $L_{\mu\nu}$  and light cone tensors.

- Starting from a Lorentz vector  $X_\mu$  show that you can define light cone coordinates  $(X_+, X_-, X_i)$ , in terms of two null vectors  $n_\pm^\mu$ .
- Show that the equality of components of the Lorentz tensors  $A_{\mu_1\mu_2\dots\mu_n} = B_{\mu_1\mu_2\dots\mu_n}$  implies the equality of the light cone tensor components, i.e.

$$A_{++++} = B_{++++}, \quad A_{++--} = B_{++--}, \quad \dots \quad A_{----} = B_{----},$$

and give explicitly the light cone components  $L_{++}$ ,  $L_{+-}$ ,  $L_{-+}$ ,  $L_{--}$  in terms of the components of a Lorentz 2-tensor  $L_{\mu\nu}$ .

- Furthermore, show that the trace of a rank-2 light cone tensor  $A$  is given by

$$A^\mu{}_\mu = -A_{+-} - A_{-+} + A_{ii}.$$

## 2. Light cone gauge and mode expansion (intermediate)

Using our newly found knowledge about light cone tensors, we will investigate the form of the angular momentum generator. The mode expansion is given by

$$X^\mu(\sigma, \tau) = x_0^\mu + \kappa^2 p^\mu \tau + \frac{i\kappa}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in(\tau-\sigma)} + \frac{i\kappa}{\sqrt{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\tau+\sigma)}.$$

- In the earlier problem 2.1.a) you derived the angular momentum current  $\mathcal{J}_{\mu\nu}^\alpha$  and its conserved charge

$$J_{\mu\nu} = \int_0^{2\pi} d\sigma \mathcal{J}_{\mu\nu}^0.$$

Express  $\mathcal{P}_\mu^0$  in terms of the mode expansion. Calculate  $J_{\mu\nu}$  in terms of the mode expansion. *Hint:* Contemplate the meaning of *conserved* quantity and use

$$\int_0^{2\pi} d\sigma e^{in\sigma} = 2\pi \delta_{n,0}.$$

- Express  $J_{-i}$  in terms of the above derived mode expansion for the Lorentz tensor  $J_{\mu\nu}$ . In a quantum field theory, symmetry generators should be realised by hermitian operators

$$(J_{\mu\nu})^\dagger = J_{\mu\nu}.$$

Assume canonical commutation relations  $[x^\mu, p^\nu] = i\eta^{\mu\nu}$ , and show that  $J_{-i}$  is *not* hermitian. “Hermiticise” the generator.

→

### 3. Maxwell and Kalb-Ramond fields (intermediate)

Light cone gauge is not only useful in string theory to extract physical information. It also is a valid gauge in other theories. First we will work on the Maxwell gauge field  $A_\mu$ . Then we will turn to the Kalb-Ramond field  $B_{\mu\nu}$  which will enter our description of quantum string theory in due course. (If you feel confident enough you can skip parts a) and b). Otherwise work carefully through all the subproblems for maximal benefit! You may find chapter 10 in Zwiebach – “A first course in String theory” useful. Use the language of differential forms if you are familiar with it.)

- a) The Maxwell field  $A_\mu(x)$  has a gauge symmetry

$$A'_\mu = A_\mu + \partial_\mu \epsilon(x).$$

We define the antisymmetric field strength tensor  $F_{\mu\nu}$  by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Show that  $F_{\mu\nu}$  is gauge invariant and derive the equations of motion for  $A_\mu$  from the action (leaving coupling constants aside)

$$S_{\text{YM}} = -\frac{1}{4} \int d^D x F_{\mu\nu} F^{\mu\nu}.$$

Rewrite the equations of motion in momentum space  $\partial_\mu \leftrightarrow p_\mu$ .

- b) We want to implement light cone gauge. Express the gauge transformation in momentum space. Show that, by a sensible choice of  $\epsilon(p)$ , you can gauge away the  $+$ -component of the light cone gauge field ( $A_+, A_-, A_i$ ) and deduce that the equation of motion in momentum space drastically simplifies in this gauge. Count the total number of *independent* degrees of freedom of the gauged Maxwell field.

The Kalb-Ramond field  $B_{\mu\nu}$  is an antisymmetric Lorentz tensor with the gauge symmetry transformation

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu.$$

We define a field strength and an action for  $B_{\mu\nu}$  by

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \quad \text{and} \quad S_{\text{KR}} = -\frac{1}{12} \int d^D x H_{\mu\nu\rho} H^{\mu\nu\rho}.$$

- c) Show that the gauge transformation of  $B_{\mu\nu}$  has a redundancy

$$\epsilon'_\mu = \epsilon_\mu + \partial_\mu \lambda$$

under which  $B_{\mu\nu}$  is invariant. Express the gauge transformations in light cone momentum space and show that you can gauge away the component  $\epsilon_+$ , such that the effective gauge transformation of  $B_{\mu\nu}$  is generated by  $\epsilon_-$  and  $\epsilon_i$ .

- d) Go through the steps in a), b) for  $B_{\mu\nu}$  and  $H_{\mu\nu\rho}$  – bearing in mind the result of c) – and show that the Kalb-Ramond field has only one independent degree of freedom in four dimensions.
- e) (advanced) In four dimensions, we can define a “dual field”  $\bar{H}_\mu$  by contracting the field strength  $H^{\mu\nu\rho}$  with the totally antisymmetric tensor of fourth order

$$\bar{H}_\mu = \varepsilon_{\mu\nu\rho\kappa} H^{\nu\rho\kappa}.$$

Using the result you found in the last part of this problem show that the dual field can be expressed by the derivative of a single scalar field. What does this imply for the Kalb-Ramond field in four dimensions?

## 1. Virasoro algebra (intermediate)

In this exercise we want to investigate in detail the Virasoro algebra as it appears up in light cone string theory. For simplicity we will only work with the left movers  $L_n^L$  as the right movers  $L_n^R$  commute with these and satisfy an identical algebra. The mode operators  $\alpha_n^i$  (with  $i = 1, \dots, D - 2$ ) satisfy the algebra (we drop the L/R superscript)

$$[\alpha_n^i, \alpha_m^j] = m\delta^{ij}\delta_{n+m},$$

and the *normal ordered* Virasoro generators are given by

$$L_n = \frac{1}{2} \sum_{p \geq 0} \alpha_{n-p}^i \alpha_p^i + \frac{1}{2} \sum_{p < 0} \alpha_p^i \alpha_{n-p}^i.$$

An algebra  $\mathfrak{g}$  is a Lie algebra if its product  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  (called the Lie bracket) is antisymmetric  $[a, b] = -[b, a] \forall a, b \in \mathfrak{g}$  and satisfies the Jacobi identity

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \forall a, b, c \in \mathfrak{g}.$$

a) Show that the commutator of two Virasoro generators with  $m + n \neq 0$  is given by

$$[L_m, L_n] = \frac{1}{2} \sum_p p \alpha_{m-p}^i \alpha_{p+n}^i + (m - p) \alpha_{n+m-p}^i \alpha_p^i$$

b) By relabelling the summands, rewrite the above result in the following form

$$[L_m, L_n] = (m - n)L_{n+m}.$$

Argue that the complete solution, including the terms  $n = -m$  is given by

$$[L_m, L_n] = (m - n)L_{m+n} + C(m)\delta_{m+n,0}$$

where  $C(m)$  is a real valued, odd ( $C(-m) = -C(m)$ ) function. The last term is called the *central extension* of the Virasoro algebra. Determine  $C(m)$  up to two constants by considering the Jacobi identity. The solution is given by  $C(m) = \frac{1}{12}(D - 2)(m^3 - m)$ .

→

## 2. Analytical continuation of the $\zeta$ -function (intermediate – hard)

In the lecture you have seen the peculiar result of the sum  $\zeta(-1) = \sum_{n=1}^{\infty} n = -\frac{1}{12}$ . In this exercise we will try to understand where the result comes from by analytically continuing the  $\zeta$ -function. The  $\Gamma$  and  $\zeta$  functions of a complex variable  $z$  are given by

$$\Gamma(z) = \int_0^{\infty} dt e^{-t} t^{z-1} \quad \text{and} \quad \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

- a) We start by regularising  $\zeta(-1)$  using a small parameter  $\epsilon$ . Show that we can write the zeta function like  $\zeta_{\epsilon}(-1) = -\frac{\partial}{\partial \epsilon} \sum_{n=1}^{\infty} e^{-n\epsilon}$  in the limit  $\epsilon \rightarrow 0$ . Argue that the sum in this expression is convergent and give the solution. Expand the expression for small  $\epsilon$  and show that the result is given by  $\zeta_{\epsilon}(-1) \approx \frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)$ .
- b) Show that for  $\text{Re}(z) > 1$  you can write

$$\Gamma(z)\zeta(z) = \int_0^{\infty} \frac{dt t^{z-1}}{e^t - 1}.$$

Conclude that it is possible to rewrite the integral to give

$$\begin{aligned} \Gamma(z)\zeta(z) &= \int_0^1 dt t^{z-1} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12} \right) + \frac{1}{z-1} - \frac{1}{2z} + \frac{1}{12(z+1)} \\ &\quad + \int_1^{\infty} \frac{dt t^{z-1}}{e^t - 1}. \end{aligned}$$

- c) (advanced) The right hand side is well defined for  $\text{Re}(z) > -2$  (why?). We know that  $\Gamma(z)$  has poles for  $z = 0, -1, -2, \dots$  with residues

$$\text{Res}_{z_0=-n}[\Gamma(z_0)] = \frac{(-1)^n}{n!}.$$

Conclude that the values of  $\zeta(z)$  at  $z = 0$  and  $z = -1$  are

$$\zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-1) = -\frac{1}{12}.$$

## 3. Poincaré transformations (easy)

Poincaré transformations  $x^{\mu} \mapsto \Lambda^{\mu}_{\nu} x^{\nu} + a^{\mu}$  form a group whose product is defined as  $T(\Lambda_1, a_1)T(\Lambda_2, a_2) = T(\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2)$ . The inverse reads  $T(\Lambda, a)^{-1} = T(\Lambda^{-1}, -\Lambda^{-1}a)$ . Consider an infinitesimal transformation with generators  $\mathcal{J}$  and  $\mathcal{P}$

$$T(1 + \omega, \epsilon) = 1 + \frac{i}{2} \omega^{\mu\nu} \mathcal{J}_{\mu\nu} - i\epsilon^{\mu} \mathcal{P}_{\mu} + \dots$$

They define the Lie algebra of the Poincaré group. Show that

$$T(\Lambda, a)T(1 + \omega, \epsilon)T(\Lambda, a)^{-1} = T(\Lambda(1 + \omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a).$$

How do  $\mathcal{J}$  and  $\mathcal{P}$  transform under  $T(\Lambda, a)$ ? What relations do you get when you take  $\Lambda, a$  to be infinitesimal as well?



## 1. Lorentz invariance in light cone gauge (hard)

Since Lorentz invariance is obscured due to the light cone gauge it is not obvious that the following commutator vanishes

$$[\mathcal{J}^{-i}, \mathcal{J}^{-k}] \stackrel{?}{=} 0.$$

This exercise sheet will be solely concerned with the calculation of this commutator and its physical implications for bosonic string theory. In a previous sheet the generator was determined to be given by (up to a doubling of the latter term due to left and right movers)

$$\mathcal{J}^{-i} = \frac{1}{2}(p^- x_0^i + x_0^i p^-) - x_0^- p^i + i \sum_{n=1}^{\infty} (\alpha_{-n}^- \alpha_n^i - \alpha_{-n}^i \alpha_n^-).$$

Furthermore, we have

$$p^- = \frac{L_0}{2\kappa^2 p^+} \quad \text{and} \quad \alpha_n^- = \frac{\sqrt{2}}{\kappa p^+} L_n$$

with  $L_n$  the Virasoro generators from the previous sheets. We depart here from the definition of sheet 4 and account for the normal ordering ambiguity in  $L_0$  by defining

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i - a$$

where  $a$  is the so called intercept. This means we have two free parameters to adjust during this calculation: the intercept  $a$  and the dimension  $D$  of spacetime. *There are many subtleties in this calculation. Careful checks after every step are recommended.*

- a) Begin by calculating all the possible commutators between the zero modes  $p^+, p^-, p^i, x_0^-, x_0^i$  and the  $\alpha_n^i, \alpha_n^-$  modes using the given commutators

$$[p^+, x_0^-] = i, \quad [p^j, x^k] = -i\delta^{jk}, \quad [\alpha_m^i, \alpha_n^j] = m\delta_{m+n}\delta^{ij}.$$

*Hint:* Watch out for subtleties when it comes to commutators with  $p^-$  and  $\alpha^-$ . The commutator  $[\alpha_m^-, \alpha_n^-]$  is the hardest here. However, you know its naive form already from the last sheet. Remember that there is a normal ordering ambiguity for  $\alpha_0^-$ !

- b) Calculate the commutator. The expected result is

$$\begin{aligned} [\mathcal{J}^{-i}, \mathcal{J}^{-j}] = & 2 \left( p^- - \frac{\sqrt{2}}{\kappa} \alpha_0^- \right) \frac{1}{p^+} \sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n}^{[i} \alpha_n^{j]} \\ & + \frac{2}{\kappa^2 (p^+)^2} \sum_{n=1}^{\infty} \left( \left[ \frac{D-2}{12} - 2 \right] n + \frac{1}{n} \left[ 2a - \frac{D-2}{12} \right] \right) \alpha_{-n}^{[i} \alpha_n^{j]}. \end{aligned}$$

For general values of  $a$  and  $D$  we say that the symmetry is *anomalous* because the right hand side is not zero. However, we can make it vanish. What are the reasons for the first term and conditions for the second term in this expression to vanish?

## 1. Kaluza-Klein theory (warm-up)

Already as early as 1921 T. Kaluza proposed a model to unify gravity and electromagnetism by introducing a (large) fifth dimension. O. Klein greatly improved this ansatz by introducing the idea of a compact fifth dimension. In Kaluza-Klein theory the metric tensor  $G_{MN}$  in  $4+1$  dimensions is split into 3 fields: a metric field  $g_{\mu\nu}$  in  $3+1$  dimensions, a  $U(1)$  gauge field  $A_\mu$  and a scalar called the *dilaton*  $\phi$ .

- a) Begin by considering a massless scalar field  $\Phi$  in  $D$  dimensions. Argue that compactifying one spacelike dimension on a circle of radius  $R$ :

$$x^{D-1} \sim x^{D-1} + 2\pi R$$

will give you an infinite tower of massive scalar fields in  $D - 1$  dimensions and one massless scalar. *Hint:* Contemplate the kind of momenta you get in compact spaces. Use the Fourier series. Write down the equations of motion of the scalar in  $D$  dimensions to see which  $D - 1$  dimensional modes are massive and which massless.

- b) How can you make this tower of massive scalar fields vanish or at least impossible to detect at small energies?

- c) (optional) Now we return to KK theory. The metric has the following form

$$G_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} - \kappa^2 \phi A_\mu A_\nu & -\kappa \phi A_\mu \\ -\kappa \phi A_\nu & \phi \end{pmatrix}.$$

5D gravity is of course invariant under reparametrisations. However after singling out the fifth dimension we can only have 4D reparametrisations and a transformation

$$x^\mu = \tilde{x}^\mu \quad \text{and} \quad x^5 + f(x^\mu) = \tilde{x}^5.$$

Show that this transformation implies a gauge transformation on the field  $A_\mu$  ( which is contained in  $G_{\mu 5}$ ).

We conclude by remarking that in KK theory the compactification of one dimension yields one massless metric field, one massless gauge field and one massless scalar along with three infinite towers of massive modes. Let us now turn to string theory.

→

## 2. T-duality: Self-dual radius (intermediate)

When compactifying one dimension of string theory on a circle, new stringy effects appear that cannot be seen in field theory. One of these effects is *winding* of the string around the compactified dimension. We compactify the coordinate  $D - 2$  on a circle (drop the index  $D - 2$  for simplicity) and request that

$$X^{D-2}(\sigma + 2\pi, \tau) = X^{D-2}(\sigma, \tau) + 2\pi\kappa^2 w$$

where  $w = mR/\kappa^2$  is the *winding*. The left and right movers are then defined as usual for all directions except for the compact dimension where

$$X_L = \frac{1}{2}x + \frac{\kappa^2}{2}(p + w)\xi^L + \frac{i\kappa}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha_n^L}{n} \exp(-in\xi^L),$$

$$X_R = \frac{1}{2}x + \frac{\kappa^2}{2}(p - w)\xi^R + \frac{i\kappa}{\sqrt{2}} \sum_{n \neq 0} \frac{\alpha_n^R}{n} \exp(-in\xi^R).$$

- a) Derive the  $(D - 1)$ -dimensional mass-squared  $M^2 = -p^2$  of states in the presence of a compact dimension in terms of the level operators  $N^L$  and  $N^R$ , the winding number  $m$  and compact momentum  $p = n/R$ . *Hint:* Use the Virasoro constraints  $L_0^L = L_0^R = a$ .
- b) Show that the level matching constraint  $N^L = N^R$  does not hold for strings with both winding number  $m$  and Kaluza-Klein momentum number  $n$  not equal zero. What happens to these states at  $R \rightarrow \infty$ ?
- c) Consider the mass formula for the cases

$$m = n = 0; \quad m = 0, n \neq 0; \quad m \neq 0, n = 0; \quad m = n = \pm 1; \quad m = -n = \pm 1.$$

For which values of  $N^L$  and  $N^R$  does the spectrum (possibly) contain tachyonic and massless states? What is their spin (scalar, vector, tensor) as viewed from  $D - 1$  non-compact spacetime dimensions?

- d) In the case of non-zero winding and non-zero compact momentum ( $m = n = \pm 1$  and  $m = -n = \pm 1$ ) show that there is a special radius  $R^*$  where some states become massless. What happens for the other cases at this radius?

Nota bene: What happened here? After compactifying one dimension we were left with two massless vector states with  $U(1) \times U(1)$  gauge group, that is one more than in KK theory. Choosing the self-dual radius  $R^*$  produces four additional massless vectors which combine with the generic ones to  $SU(2) \times SU(2)$  gauge fields. What we witness here is symmetry enhancement at the selfdual point! These are true string theory effects that cannot appear in KK theory.

## 1. $p$ -branes (intermediate)

The mechanics of a generic  $p$ -brane (in the sense of an object with  $p + 1$  dimensional worldvolume) can be described by the Dirac-type action

$$S = -T \int d^{p+1}\xi \sqrt{-\det \gamma},$$

where we have  $p + 1$  coordinates  $\xi^\alpha$  with  $\alpha = 0, \dots, p$ . The matrix  $\gamma_{\alpha\beta}$  is the *pullback* of the Minkowski metric onto the brane

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} \eta_{\mu\nu}.$$

Show that this action is equivalent to the Polyakov-style action

$$S = -\frac{T}{2} \int d^{p+1}\xi \sqrt{-\det g} [g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - (p - 1)],$$

where  $g_{\alpha\beta}$  is the dynamical worldvolume metric. Do these actions look familiar to you in the cases  $p = 0, 1$ ?

## 2. Stretched strings (intermediate)

An open string can stretch between two  $Dp$ -branes. In fact there are four possibilities for a string to stretch between two branes, called *sectors*. Two of the sectors are strings beginning and ending on the same branes denoted [11] and [22]. The sectors [12] and [21] contain the cases where the string stretched between the branes. The last two cases are different because orientation matters. We will be interested in the case where the endpoints of the string lie on two different branes.

- a) Write down the mode expansion for a string stretched between two parallel  $Dp$ -branes and interpret the result.
- b) How does the distance between the branes affect the spectrum of the string? What happens for coincident branes? *Hint*: Consider the mass-squared.

→

### 3. Orbifolds (intermediate – hard)

After having seen how a string behaves under compactification of one dimension, we want to find out how it behaves under restricting it to the half line  $x^{D-1} \geq 0$  by the identification

$$x^{D-1} \sim -x^{D-1}.$$

Such a space is called *orbifold* in string theory. Again we abbreviate the relevant coordinate  $X(\xi) := X^{D-1}(\xi)$ , and introduce an operator  $U$  acting as ( $\mu = 0, \dots, D-2$ )

$$UX(\sigma)U^{-1} = -X(\sigma + \pi), \quad UX^\mu(\sigma)U^{-1} = X^\mu(\sigma + \pi).$$

$U$  is a symmetry of the orbifold theory, so only states invariant under  $U$  are physical.

- a) How does  $U$  act on the modes  $x$ ,  $p$ ,  $\alpha_n^L$  and  $\alpha_n^R$  of the coordinate  $X$ ?
- b) Define the string vacuum  $|0; q^\mu, r\rangle$  where  $\mu = 0, \dots, D-2$  and  $r$  is the momentum in the folded dimension. We assume that

$$U|0; q^\mu, 0\rangle = |0; q^\mu, 0\rangle.$$

Give the action of  $U$  on  $|0; q^\mu, r\rangle$  and write down the ground states of the orbifold theory.

- c) What are the massless states of the orbifold theory?

**1. Two-point function** (easy)

In this exercise we want to compute the closed string propagator

$$\langle X^\mu(z, \bar{z}) X^\nu(z', \bar{z}') \rangle = -\frac{\kappa^2}{2} \eta^{\mu\nu} \log |z - z'|^2$$

which is given by the difference of the time-ordered and the normal ordered product of the operators  $X^\mu(z, \bar{z})$  and  $X^\nu(z', \bar{z}')$ . Assume  $|z| > |z'|$ . *Hint:* You may ignore the effects of the centre of mass coordinates  $x^\mu$  or use  $:p^\mu x^\nu: = x^\nu p^\mu$ .

**2. Conformal transformations** (intermediate)

Consider conformal transformations  $z \rightarrow z'(z)$ . Primary fields transform as tensors under conformal transformations

$$\mathcal{O}'(z, \bar{z}) = \left( \frac{\partial z'}{\partial z} \right)^h \left( \frac{\partial \bar{z}'}{\partial \bar{z}} \right)^{\bar{h}} \mathcal{O}(z'(z), \bar{z}'(\bar{z}))$$

- a) How does a primary field transform under infinitesimal transformations  $z' \rightarrow z + \zeta(z)$ ?
- b) Show that the operator  $:e^{ikX}$ : for a single scalar field  $X$  is primary by computing its OPE with the stress-energy tensor. Determine the conformal weights  $h$  and  $\bar{h}$ .

**3. The complex logarithm** (intermediate)

The propagator that you calculated above and the two-point function

$$\langle \partial X^\mu(z, \bar{z}) \bar{\partial} X^\nu(z', \bar{z}') \rangle = \pi \kappa^2 \eta^{\mu\nu} \delta^2(z - z', \bar{z} - \bar{z}')$$

are related by the two derivatives. Show that

$$\partial \bar{\partial} \log |z|^2 = 2\pi \delta^2(z, \bar{z})$$

- a) ... by considering the divergence theorem.
- b) ... by regulating the singularity at  $z = 0$ .

→

#### 4. Free fermions (intermediate)

In QFT, fermions are represented by *Grassmann-valued* fields. In contradistinction to ordinary numbers, Grassmann numbers  $\eta_1, \eta_2$  *anticommute* with each other

$$\eta_1\eta_2 = -\eta_2\eta_1.$$

For instance, the Pauli exclusion principle is realised by the anticommuting Grassmann numbers. We introduce a free real fermion field in two dimensions with the action

$$S = \frac{1}{2\pi} \int d^2z \psi \bar{\partial}\psi + \bar{\psi} \partial\bar{\psi}.$$

The field has the OPE

$$\psi(z)\psi(w) = -\psi(w)\psi(z) = \frac{1}{z-w} + \dots$$

and similar for  $\bar{\psi}$ . The strange looking minus sign is due to the anti-commutative nature of Grassmann numbers. The stress-energy tensor for the free fermion reads

$$T(z) = -\frac{1}{2}:\psi\partial\psi:.$$

- a) Show that  $\psi$  is a primary field with weight  $(\frac{1}{2}, 0)$ .
- b) Compute the OPE of two  $T$ 's and give the central charge of the theory.

## 1. Veneziano Amplitude (intermediate – hard)

The Veneziano amplitude<sup>1</sup> led to the discovery of string theory. In this problem we will attempt to calculate this amplitude. The open string tachyon vertex operator is given by an integral over the boundary of the string

$$V(k) = \sqrt{g_s} \int dx : \exp(ik_\mu X^\mu(x)) :$$

such that the four tachyon scattering amplitude  $A_4(k_1, k_2, k_3, k_4)$  is given by the expression

$$A_4(k_1, k_2, k_3, k_4) \sim \frac{1}{g_s} \langle V_1 \dots V_4 \rangle \sim g_s \int \prod_{x_i < x_{i+1}} dx_i \langle : e^{ik_1 \cdot X(x_1)} : \dots : e^{ik_4 \cdot X(x_4)} : \rangle.$$

The ordering of insertions  $x_i$  is due to Chan–Paton factors.

- a) The expectation value is computed using Wick’s theorem and the two-point correlator

$$\langle X^\mu(x) X^\nu(y) \rangle = -2\kappa^2 \eta^{\mu\nu} \log |x - y|.$$

A generic Wick contraction will have  $n_{ij}$  correlators between the pair of points  $(x_i, x_j)$ . Count the combinatorial number of contractions that lead to a configuration specified by the numbers  $n_{ij}$ . *Hint:* You need to split  $n_i = \sum_j n_{ij}$  fields at point  $x_i$  into  $n_{ij}$  contractions to point  $x_j$ .

- b) Collect the combinatorial factors and expansion coefficients of  $e^{ik_i \cdot X(x_i)}$ . Then perform the sum over  $n_{ij}$ ’s. *Hint:* You should obtain the following integral expression for  $A_4$  (up to the momentum-conserving delta function which is more subtle)

$$A_4 \sim g_s \delta^{26} \left( \sum_i k_i \right) \int \prod_{i=1}^4 dx_i \prod_{j < l} |x_j - x_l|^{2\kappa^2 k_j \cdot k_l}.$$

- c) Show that the integral is invariant under the  $SL(2, \mathbb{R})$  Möbius transformation

$$x_i \rightarrow \frac{ax_i + b}{cx_i + d}$$

for on-shell momenta  $k_i^2 = \kappa^{-2}$ . *Hint:* Use momentum conservation  $\sum_i k_i = 0$ .

- d) The integral given above is divergent because it has the non-compact Möbius group as a symmetry. It thus contains an irrelevant factor of the group volume which is infinite. We divide by the latter and use the symmetry to set  $x_1 = 0$ ,  $x_2 = x$ ,  $x_3 = 1$  and  $x_4 \rightarrow \infty$ . Explain why the amplitude after the transformation reduces to

$$A_4 \sim g_s \delta^{26} \left( \sum_i k_i \right) \int dx |x|^{2\kappa^2 k_1 \cdot k_2} |1 - x|^{2\kappa^2 k_2 \cdot k_3} + (k_2 \leftrightarrow k_3).$$

What is the integration range of  $x_2$  now? Why? What happened to the normalisation in front of the integral?

→

<sup>1</sup>For the original paper see <http://dx.doi.org/10.1007/BF02824451>



e) The resulting integral is well known. It is in the form of the Euler beta function

$$B(a, b) = \int_0^1 dy y^{a-1} (1-y)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Write down the solution for the amplitude. *Hint:* Use the Mandelstam variables

$$s = -(k_1 + k_2)^2, \quad t = -(k_1 + k_3)^2, \quad u = -(k_1 + k_4)^2$$

to simplify the result.

f) Where does this amplitude have poles? What do these poles correspond to?

## 2. Schwarzian derivative (intermediate)

The stress-energy tensor transforms under a finite conformal transformation  $z \rightarrow z' = f(z)$  as

$$T(z) \rightarrow T'(z) = (\partial f)^2 T(z') + \frac{c}{12} \mathcal{S}(z', \bar{z}')$$

where

$$\mathcal{S}(z', \bar{z}') = \frac{\partial f(z) \partial^3 f(z) - \frac{3}{2} (\partial^2 f(z))^2}{(\partial f)^2}$$

is the Schwarzian derivative.

- a) Show that the Schwarzian derivative reproduces the correct infinitesimal transformation.
- b) Show that the Schwarzian derivative has the correct property under successive conformal transformations.
- c) Prove that for a transformation

$$f(z) = \frac{az + b}{cz + d}$$

the Schwarzian derivative yields  $\mathcal{S}(z, \bar{z}) = 0$ . Why is this not surprising?

**1. Regge behaviour and hard scattering limit** (intermediate – hard)

We want to have a look at the features of the Veneziano amplitude

$$A_4 \sim g_s \delta^{26} \left( \sum_i k_i \right) (I(s, t) + I(t, u) + I(u, s))$$

where

$$I(s, t) = \frac{\Gamma(-1 - s\kappa^2)\Gamma(-1 - t\kappa^2)}{\Gamma(-2 - s\kappa^2 - t\kappa^2)}$$

obtained on the last problem sheet by taking two interesting limits.

- a) The first limit we want to take is  $s \rightarrow \infty$  with  $t$  fixed. This is the so called *Regge limit*. Why does this correspond to high energy and small angle scattering? Show that in this limit  $I(s, t)$  reduces to

$$I(s, t) \sim s^{1+\kappa^2 t} \Gamma(-1 - \kappa^2 t).$$

*Hint:* Use Stirling's approximation of the  $\Gamma$  function  $\Gamma(1 + x) \approx x^x e^{-x} \sqrt{2\pi x}$ .

- b) The second limit is the hard scattering limit  $s \rightarrow \infty$  with  $t/s$  fixed. Argue that this corresponds to high energy and fixed angle scattering. Show that the amplitude reduces to

$$A_4 \sim \exp(-s \log(\kappa^2 s) - t \log(\kappa^2 t) - u \log(\kappa^2 u)).$$

*Hint:* It might be easier to look at the integral expression of the amplitude and attempt a saddle point approximation, but the limit can also be taken straightforwardly.

→

## 2. Low-energy effective action (easy – intermediate)

In the string frame the low-energy effective action is given by

$$S = \frac{1}{2\kappa^2} \int d^{26} X \sqrt{-\det G(X)} e^{-2\Phi} \left( R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4\partial_\mu \Phi \partial^\mu \Phi \right).$$

Here  $G_{\mu\nu}$  is the metric,  $R$  the associated Ricci scalar,  $H_{\mu\nu\lambda} = 3\partial_{[\mu} B_{\nu\lambda]}$  is the Kalb-Ramond field strength and  $\Phi$  is a scalar, the dilaton field.

- a) Show that the equations of motion of these fields are equivalent to the vanishing of the  $\beta$  functions

$$\begin{aligned} \beta_{\mu\nu}(G) &= \kappa^2 R_{\mu\nu} + 2\kappa^2 \nabla_\mu \nabla_\nu \Phi - \frac{\kappa^2}{4} H_{\mu\lambda\sigma} H_\nu{}^{\lambda\sigma}, \\ \beta_{\mu\nu}(B) &= -\frac{\kappa^2}{2} \nabla^\lambda H_{\lambda\mu\nu} + \kappa^2 \nabla^\lambda \Phi H_{\lambda\mu\nu}, \\ \beta(\Phi) &= -\frac{\kappa^2}{2} \nabla^2 \Phi + \kappa^2 \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\kappa^2}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}. \end{aligned}$$

- b) The kinetic energy term of the dilaton in the action seems to have the wrong sign. Explain why this is not so.

### 1. Linear dilaton (intermediate)

The general worldsheet action for massless background fields is given by

$$S = \frac{1}{4\pi\kappa^2} \int d^2\xi \sqrt{-\det g} \left( (g^{\alpha\beta} G_{\mu\nu} + \varepsilon^{\alpha\beta} B_{\mu\nu}) \partial_\alpha X^\mu \partial_\beta X^\nu + \kappa^2 R\Phi \right).$$

To consider a concrete example of a background for string theory we want to have a look at the *linear dilaton background* where

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = 0 \quad \text{and} \quad \Phi = V_\mu X^\mu$$

with  $V_\mu$  a constant vector and  $R$  the worldsheet Ricci scalar.

- a) Show that the  $\beta$  functions defined on the last problem sheet vanish for  $V_\mu V^\mu = (26 - D)/6\kappa^2$ .
- b) Derive the worldsheet energy-momentum tensor

$$T(z) = -\frac{1}{\kappa^2} : \partial X^\mu \partial X_\mu : + V_\mu \partial^2 X^\mu$$

of this theory and show that the central charge is given by

$$c = D + 6\kappa^2 V_\mu V^\mu.$$

### 2. (D)BI action (intermediate – hard)

In an attempt to solve the problem of the infinite classical self-energy of a charged point particle Born and Infeld proposed a non-linear generalisation of Maxwell's theory

$$S_{BI} \approx \int d^4x \sqrt{-\det(\eta_{\alpha\beta} + kF_{\alpha\beta})}$$

with  $k$  a constant. A further generalisation of this action appears in open string theory as part of the world-volume action of a  $Dp$ -brane in the form of the Dirac-Born-Infeld action

$$S_{DBI} = -T_{Dp} \int d^{p+1}\xi \sqrt{-\det(\eta_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + kF_{\alpha\beta})}.$$

In both cases,  $F_{\alpha\beta}$  is the Maxwell field strength. For simplicity we'll have a look at the BI-action.

- a) Show that the Born-Infeld Lagrangian can be rewritten

$$\mathcal{L}_{BI} = \sqrt{\det(\eta_{\alpha\beta} - kF_{\alpha\beta})} = \exp\left(\frac{1}{4} \text{tr} \log(\eta_{\alpha\beta} - k^2(F^2)_{\alpha\beta})\right)$$

where  $(F^2)_{\alpha\beta} = F_{\alpha\gamma} F_\beta^\gamma$ .

- b) Determine the equations of motion of the gauge field from the BI action and expand them in  $k$  to derive the leading order correction to the vacuum Maxwell field equations.
- c) (advanced) Expand the DBI action to fourth order in  $k$  and show that the quadratic term gives Maxwell's action. Repeat b) for the DBI action.

**1. World-sheet supersymmetry** (intermediate)

An action with global worldsheet supersymmetry is given by

$$S = -\frac{1}{4\pi\kappa^2} \int d^2\xi (\partial^\alpha X^\mu \partial_\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu).$$

The Grassmann-valued fields  $\psi^\mu$  are two-dimensional Majorana spinors. The  $\rho^\alpha$  are  $2 \times 2$   $\gamma$ -matrices satisfying

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$$

and the Dirac conjugate is  $\bar{\psi} = i\psi^\dagger \rho^0$  with representation

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- a) Show that this action is invariant under  $\mathcal{N} = 1$  supersymmetry

$$\delta X^\mu = \bar{\epsilon} \psi^\mu, \quad \delta \psi^\mu = \rho^\alpha \partial_\alpha X^\mu \epsilon.$$

Why are the components of  $\psi$  and  $\epsilon$  Weyl “spinors”?

- b) Evaluate the commutators  $[\delta_1, \delta_2]X^\mu$  and  $[\delta_1, \delta_2]\psi^\mu$  to show that the commutator of two supersymmetry transformations amount to a translation along the world-sheet.
- c) Derive the Noether current (supercurrent) of supersymmetry transformations. Meditate on the relation between the supercurrent and the energy-momentum tensor.

**2. Grande Finale: The super-Particle** (intermediate)

A massless supersymmetric particle in  $D$ -dimensional Minkowski spacetime is described by the action

$$S_\ell = \int d\tau \left( \frac{\dot{X}^\mu \dot{X}_\mu}{2e} + \frac{i\dot{X}^\mu \psi_\mu \chi}{e} - i\psi^\mu \dot{\psi}_\mu \right)$$

where  $e$  is the einbein and  $\chi$  its fermionic partner.  $\psi^\mu$  are 2D Majorana fermions.

- a) Derive the equations of motion for  $X^\mu(\tau)$ ,  $\psi^\mu(\tau)$  as well as  $e$  and  $\chi$ . Find the correct supersymmetry transformations for  $e$  and  $\chi$  by requiring that the action is invariant under (local!) supersymmetry transformations with

$$\delta X^\mu = i\epsilon \psi^\mu \quad \text{and} \quad \delta \psi^\mu = \frac{1}{2e} (\dot{X}^\mu - i\chi \psi^\mu) \epsilon.$$

- b) Argue that you can set  $e = 1$  and  $\chi = 0$ . Write down the gauged action. Important: What are the constraint equations that follow from this procedure? Interpret them!
- c) Consider the (global) supersymmetry transformations (derive them from the ones given in a)) and check that the commutator of two supersymmetry transformations will result in a  $\tau$  translation by an amount  $\delta\tau$ .