1. Linear dilaton (intermediate)

The general worldsheet action for massless background fields is given by

$$S = \frac{1}{4\pi\kappa^2}\int d^2\xi\sqrt{-\det g}\Big((g^{\alpha\beta}G_{\mu\nu} + \varepsilon^{\alpha\beta}B_{\mu\nu})\partial_\alpha X^\mu\partial_\beta X^\nu + \kappa^2 R\Phi\Big).$$

To consider a concrete example of a background for string theory we want to have a look at the *linear dilaton background* where

$$G_{\mu\nu} = \eta_{\mu\nu}, \qquad B_{\mu\nu} = 0 \qquad \text{and} \qquad \Phi = V_\mu X^\mu$$

with V_μ a constant vector and R the worldsheet Ricci scalar.

a) Show that the β functions defined on the last problem sheet vanish for $V_\mu V^\mu = (26 - D)/6\kappa^2$.

b) Derive the worldsheet energy-momentum tensor

$$T(z) = -\frac{1}{\kappa^2}{:}\partial X^\mu\partial X_\mu{:} + V_\mu\partial^2 X^\mu$$

of this theory and show that the central charge is given by

$$c = D + 6\kappa^2 V_\mu V^\mu.$$

2. (D)BI action (intermediate – hard)

In an attempt to solve the problem of the infinite classical self-energy of a charged point particle Born and Infeld proposed a non-linear generalisation of Maxwell's theory

$$S_{BI} \approx \int d^4x\sqrt{-\det(\eta_{\alpha\beta} + kF_{\alpha\beta})}$$

with k a constant. A further generalisation of this action appears in open string theory as part of the world-volume action of a Dp-brane in the form of the Dirac-Born-Infeld action

$$S_{DBI} = -T_{\mathrm{D}p}\int d^{p+1}\xi\sqrt{-\det(\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu + kF_{\alpha\beta})}.$$

In both cases, $F_{\alpha\beta}$ is the Maxwell field strength. For simplicity we'll have a look at the BI-action.

a) Show that the Born-Infeld Lagrangian can be rewritten

$$\mathcal{L}_{BI} = \sqrt{\det(\eta_{\alpha\beta} - kF_{\alpha\beta})} = \exp\left(\frac{1}{4}\mathrm{tr}\log(\eta_{\alpha\beta} - k^2(F^2)_{\alpha\beta})\right)$$

where $(F^2)_{\alpha\beta} = F_{\alpha\gamma}F^\gamma_\beta$.

b) Determine the equations of motion of the gauge field from the BI action and expand them in k to derive the leading order correction to the vacuum Maxwell field equations.

c) (advanced) Expand the DBI action to fourth order in k and show that the quadratic term gives Maxwell's action. Repeat b) for the DBI action.