Introduction to String Theory — Problem Set 5

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1. Lorentz invariance in light cone gauge (hard)

Since Lorentz invariance is obscured due to the light cone gauge it is not obvious that the following commutator vanishes

$$[\mathcal{J}^{-i}, \mathcal{J}^{-k}] \overset{?}{=} 0\,.$$

This exercise sheet will be solely concerned with the calculation of this commutator and its physical implications for bosonic string theory. In a previous sheet the generator was determined to be given by (up to a doubling of the latter term due to left and right movers)

$$\mathcal{J}^{-i} = \frac{1}{2}(p^- x_0^i + x_0^i p^-) - x_0^- p^i + i\sum_{n=1}^{\infty}(\alpha_{-n}^- \alpha_n^i - \alpha_{-n}^i \alpha_n^-).$$

Furthermore, we have

$$p^- = \frac{L_0}{2\kappa^2 p^+} \quad \text{and} \quad \alpha_n^- = \frac{\sqrt{2}}{\kappa p^+} L_n$$

with L_n the Virasoro generators from the previous sheets. We depart here from the definition of sheet 4 and account for the normal ordering ambiguity in L_0 by defining

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}^i \alpha_n^i - a$$

where a is the so called intercept. This means we have two free parameters to adjust during this calculation: the intercept a and the dimension D of spacetime. *There are many subleties in this calculation. Careful checks after every step are recommended.*

a) Begin by calculating all the possible commutators between the zero modes p^+, p^-, p^i, x_0^-, x_0^i and the α_n^i, α_n^- modes using the given commutators

$$[p^+, x_0^-] = i, \quad [p^j, x^k] = -i\delta^{jk}, \quad [\alpha_m^i, \alpha_n^j] = m\delta_{m+n}\delta^{ij}.$$

Hint: Watch out for subtleties when it comes to commutators with p^- and α^-. The commutator $[\alpha_m^-, \alpha_n^-]$ is the hardest here. However, you know its naive form already from the last sheet. Remember that there is a normal ordering ambiguity for α_0^-!

b) Calculate the commutator. The expected result is

$$\begin{aligned}[\mathcal{J}^{-i}, \mathcal{J}^{-j}] = {} & 2\left(p^- - \frac{\sqrt{2}}{\kappa}\alpha_0^-\right)\frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}\alpha^{[i}_{-n}\alpha^{j]}{}_n \\ & + \frac{2}{\kappa^2 (p^+)^2}\sum_{n=1}^{\infty}\left(\left[\frac{D-2}{12} - 2\right]n + \frac{1}{n}\left[2a - \frac{D-2}{12}\right]\right)\alpha_{-n}^{[i}\alpha_n^{j]}.\end{aligned}$$

For general values of a and D we say that the symmetry is *anomalous* because the right hand side is not zero. However, we can make it vanish. What are the reasons for the first term and conditions for the second term in this expression to vanish?