1. Light cone tensors (easy)

We want to derive some relations between Lorentz tensors $L_{\mu\nu}$ and light cone tensors.

a) Starting from a Lorentz vector X_μ show that you can define light cone coordinates (X_+, X_-, X_i), in terms of two null vectors $n^\mu_\pm$.

b) Show that the equality of components of the Lorentz tensors $A_{\mu_1\mu_2\ldots\mu_n} = B_{\mu_1\mu_2\ldots\mu_n}$ implies the equality of the light cone tensor components, i.e.

$$A_{++\ldots+} = B_{++\ldots+}, \quad A_{++\ldots-} = B_{++\ldots-}, \quad \ldots \quad A_{--\ldots-} = B_{--\ldots-},$$

and give explicitly the light cone components L_{++}, L_{+-}, L_{-+}, L_{--} in terms of the components of a Lorentz 2-tensor $L_{\mu\nu}$.

c) Furthermore, show that the trace of a rank-2 light cone tensor A is given by

$$A^\mu_\mu = -A_{+-} - A_{-+} + A_{ii}.$$

2. Light cone gauge and mode expansion (intermediate)

Using our newly found knowledge about light cone tensors, we will investigate the form of the angular momentum generator. The mode expansion is given by

$$X^\mu(\sigma,\tau) = x^\mu_0 + \kappa^2 p^\mu \tau + \frac{i\kappa}{\sqrt{2}} \sum_{n\neq 0} \frac{\alpha^\mu_n}{n} e^{-in(\tau-\sigma)} + \frac{i\kappa}{\sqrt{2}} \sum_{n\neq 0} \frac{\tilde{\alpha}^\mu_n}{n} e^{-in(\tau+\sigma)}.$$

a) In the earlier problem 2.1.a) you derived the angular momentum current $\mathcal{J}^\alpha_{\mu\nu}$ and its conserved charge

$$J_{\mu\nu} = \int_0^{2\pi} d\sigma\ \mathcal{J}^0_{\mu\nu}.$$

Express $\mathcal{P}^0_\mu$ in terms of the mode expansion. Calculate $J_{\mu\nu}$ in terms of the mode expansion. *Hint:* Contemplate the meaning of *conserved* quantity and use

$$\int_0^{2\pi} d\sigma\, e^{in\sigma} = 2\pi\delta_{n,0}.$$

b) Express J_{-i} in terms of the above derived mode expansion for the Lorentz tensor $J_{\mu\nu}$. In a quantum field theory, symmetry generators should be realised by hermitian operators

$$(J_{\mu\nu})^\dagger = J_{\mu\nu}.$$

Assume canonical commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, and show that J_{-i} is *not* hermitian. "Hermiticise" the generator.

⟶

3. Maxwell and Kalb-Ramond fields (intermediate)

Light cone gauge is not only useful in string theory to extract physical information. It also is a valid gauge in other theories. First we will work on the Maxwell gauge field A_μ. Then we will turn to the Kalb-Ramond field $B_{\mu\nu}$ which will enter our description of quantum string theory in due course. (If you feel confident enough you can skip parts a) and b). Otherwise work carefully through all the subproblems for maximal benefit! You may find chapter 10 in Zwiebach – "A first course in String theory" useful. Use the language of differential forms if you are familiar with it.)

a) The Maxwell field $A_\mu(x)$ has a gauge symmetry

$$A'_\mu = A_\mu + \partial_\mu \epsilon(x).$$

We define the antisymmetric field strength tensor $F_{\mu\nu}$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Show that $F_{\mu\nu}$ is gauge invariant and derive the equations of motion for A_μ from the action (leaving coupling constants aside)

$$S_{\text{YM}} = -\frac{1}{4}\int d^D x\, F_{\mu\nu}F^{\mu\nu}.$$

Rewrite the equations of motion in momentum space $\partial_\mu \leftrightarrow p_\mu$.

b) We want to implement light cone gauge. Express the gauge transformation in momentum space. Show that, by a sensible choice of $\epsilon(p)$, you can gauge away the $+$-component of the light cone gauge field (A_+, A_-, A_i) and deduce that the equation of motion in momentum space drastically simplifies in this gauge. Count the total number of *independent* degrees of freedom of the gauged Maxwell field.

The Kalb-Ramond field $B_{\mu\nu}$ is an antisymmetric Lorentz tensor with the gauge symmetry transformation

$$\delta B_{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu.$$

We define a field strength and an action for $B_{\mu\nu}$ by

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \quad \text{and} \quad S_{\text{KR}} = -\frac{1}{12}\int d^D x\, H_{\mu\nu\rho}H^{\mu\nu\rho}.$$

c) Show that the gauge transformation of $B_{\mu\nu}$ has a redundancy

$$\epsilon'_\mu = \epsilon_\mu + \partial_\mu \lambda$$

under which $B_{\mu\nu}$ is invariant. Express the gauge transformations in light cone momentum space and show that you can gauge away the component ϵ_+, such that the effective gauge transformation of $B_{\mu\nu}$ is generated by ϵ_- and ϵ_i.

d) Go through the steps in a), b) for $B_{\mu\nu}$ and $H_{\mu\nu\rho}$ – bearing in mind the result of c) – and show that the Kalb-Ramond field has only one independent degree of freedom in four dimensions.

e) (advanced) In four dimensions, we can define a "dual field" $\bar{H}_\mu$ by contracting the field strength $H^{\mu\nu\rho}$ with the totally antisymmetric tensor of fourth order

$$\bar{H}_\mu = \varepsilon_{\mu\nu\rho\kappa} H^{\nu\rho\kappa}.$$

Using the result you found in the last part of this problem show that the dual field can be expressed by the derivative of a single scalar field. What does this imply for the Kalb-Ramond field in four dimensions?