10 Superstrings

Until now, encountered only bosonic d.o.f. in string theory. Matter in nature is dominantly fermionic. Need to add fermions to string theory.

Several interesting consequences:

- Supersymmetry inevitable.
- Critical dimension reduced from $D = 26$ to $D = 10$.
- Increased stability.
- Closed string tachyon absent. Stable D-branes.
- Several formulations related by dualities.

10.1 Supersymmetry

String theory always includes spin-2 gravitons. Fermions will likely include spin-$\frac{3}{2}$ gravitini → supergravity. Spacetime symmetries extended to supersymmetry.

Super-Poincaré Algebra. Super-Poincaré algebra is an extension of Poincaré algebra.

Poincaré: Lorentz rotations $M_{\mu\nu}$, translations P_μ.

$$[M, M] \sim M, \qquad [M, P] \sim P, \qquad [P, P] = 0.$$

Super-Poincaré: Odd super-translation Q^I_m (a: spinor)

$$[M, Q] \sim Q, \qquad [Q, P] = 0, \qquad \{Q^I_m, Q^J_n\} \sim \delta^{IJ}\gamma^\mu_{mn}P_\mu.$$

$\mathcal{N}$: rank of supersymmetry $I = 1, \ldots, \mathcal{N}$.

Q relates particles of

- of different spin,
- of different statistics,

and attributes similar properties to them. Symmetry between "forces" and "matter".

More supersymmetry, higher spin particles.

- gauge theory (spin ≤ 1): ≤ 16 Q's.
- gravity theory (spin ≤ 2): ≤ 32 Q's.

Superspace. Supersymmetry is symmetry of superspace. Add anticommuting coordinates to spacetime $x^\mu \to (x^\mu, \theta^a_I)$. Superfields: expansion in θ yields various fields

$$F(x, \theta) = F_0(x) + \theta^m_I F^I_m(X) + \theta^2 \ldots + \ldots + \theta^{\dim\theta}.$$

Package supermultiplet of particles in a single field.

Spinors. Representations of $Spin(D-1,1)$ (Clifford).

Complex spinors (Dirac) in $(3+1)$D belong to $\mathbb{C}^4$. Can split into chiral spinors (Weyl): $\mathbb{C}^2 \oplus \mathbb{C}^2$. Reality condition (Majorana): $\mathrm{Re}(\mathbb{C}^2 \oplus \bar{\mathbb{C}}^2) = \mathbb{C}^2$.

Spinors in higher dimensions:

- spinor dimension times 2 for $D \to D+2$.
- chiral spinors (Weyl) for D even.
- real spinors (Majorana) for $D = 0,1,2,3,4 \pmod 8$.
- real chiral spinors (Majorana–Weyl) for $D = 2 \pmod 8$.

Maximum dimensions:

- $D = 10$: real chiral spinor with 16 components (gauge).
- $D = 11$: real spinor with 32 components (gravity bound).

Super-Yang–Mills Theory. $\mathcal{N} = 1$ supersymmetry in $D = 10$ Minkowski space:

- gauge field A_μ: 8 on-shell d.o.f..
- adjoint real chiral spinor Ψ_m: 8 on-shell d.o.f..

Simple action

$$S \sim \int d^{10}x \,\mathrm{tr}\bigl(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \gamma^{\mu}_{mn}\Psi^n D_\mu \Psi^n\bigr).$$

Supergravity Theories. Four relevant models:

- $\mathcal{N} = 1$ supergravity in 11D: M-Theory.
- $\mathcal{N} = (1,1)$ supergravity in 10D: Type IIA supergravity.
- $\mathcal{N} = (2,0)$ supergravity in 10D: Type IIB supergravity.
- $\mathcal{N} = (1,0)$ supergravity in 10D: Type I supergravity.

Fields always 128+128 d.o.f. (type I: half, SYM only 8+8):

type	gr.	[4]	[3]	[2]	[1]	sc.	gravitini	spinors
M	1	0	1	0	0	0	1	0
IIA	1	0	1	1	1	1	(1,1)	(1,1)
IIB	1	1	0	2	0	2	(2,0)	(2,0)
I	1	0	0	1	0	1	(1,0)	(1,0)
SYM	0	0	0	0	1	0	0	(0,1)

M-theory has no 2-form and no dilaton: no string theory. Type IIA, IIB and I have 2-form and dilaton: strings?!

10.2 Green–Schwarz Superstring

Type II string: Add fermions Θ^m_I to worldsheet. Equal/opposite chirality: IIB/IIA

Action. Supermomentum $\Pi^\mu_\alpha = \partial_\alpha X^\mu + \delta^{IJ}\gamma^\mu_{mn}\Theta^m_I\partial_\alpha\Theta^n_J$.

$$S \sim \int d^2\xi\,\sqrt{-\det g}\,g^{\alpha\beta}\eta_{\mu\nu}\Pi^\mu_\alpha\Pi^\nu_\beta + \int\Big(\big(\Theta^1\gamma_\mu d\Theta^1 - \Theta^2\gamma_\mu d\Theta^2\big)\,dX^\mu + \Theta^1\gamma_\mu d\Theta^1\Theta^2\gamma^\mu d\Theta^2\Big).$$

Action has kappa symmetry (local WS supersymmetry). Only in $D = 10$!

Note: fermions Θ have first and second class constraints. Non-linear equations of motion. In general difficult to quantise canonically. Conformal gauge does not resolve difficulties.

Light-Cone Gauge. Convenient to apply light-cone gauge. Simplifies drastically: quadratic action, linear e.o.m.

$$S \sim \int d^2\xi\,\Big(\partial_{\mathrm{L}}\vec{X}\cdot\partial_{\mathrm{R}}\vec{X} + \tfrac{1}{2}\Theta_1\cdot\partial_{\mathrm{R}}\Theta_1 + \tfrac{1}{2}\Theta_2\cdot\partial_{\mathrm{L}}\Theta_2\Big)$$

Bosons $\vec{X}$ with $\partial_{\mathrm{L}}\partial_{\mathrm{R}}\vec{X} = 0$

- Vector of transverse $SO(8)$: $\mathbf{8}_{\mathrm{v}}$
- Left and right moving d.o.f.

Fermions Θ_1, Θ_2 with $\partial_{\mathrm{R}}\Theta_1 = 0$ and $\partial_{\mathrm{L}}\Theta_2 = 0$

- Real chiral spinor of transverse $SO(8)$: $\mathbf{8}_{\mathrm{s}}$ or $\mathbf{8}_{\mathrm{c}}$. Equal/opposite chiralities for IIB/IIA: $\mathbf{8}_{\mathrm{s}} + \mathbf{8}_{\mathrm{s}}$ or $\mathbf{8}_{\mathrm{s}} + \mathbf{8}_{\mathrm{c}}$
- Left and right moving d.o.f. in Θ_1 and Θ_2, respectively.

Spectrum. Vacuum energy and central charge:

- 8 bosons and 8 fermions for L/R: $a_{\mathrm{L/R}} = 8\zeta(1) - 8\zeta(1) = 0$. no shift a for L_0 constraint. Level zero is massless! No tachyon!
- $c = 10 + 32\frac{1}{2} = 26$ (fermions count as $\frac{1}{2}$ due to kappa).
- Super-Poincaré anomaly cancels.

Expansion into bosonic modes α_n and fermionic modes β_n. $n < 0$: creation, $n = 0$: zero mode, $n > 0$: annihilation.

Zero modes and vacuum:

- α_0 is c.o.m. momentum: $\vec{q}$.
- β_0 transforms the vacuum state:

$$\begin{array}{lll} \beta \text{ chiral } (\mathbf{8}_{\mathrm{s}}): & \mathbf{8}_{\mathrm{v}} \leftrightarrow \mathbf{8}_{\mathrm{c}} & \text{vacuum} \to |\mathbf{8}_{\mathrm{v}} + \mathbf{8}_{\mathrm{c}}, q\rangle \\ \beta \text{ anti-chiral } (\mathbf{8}_{\mathrm{c}}): & \mathbf{8}_{\mathrm{v}} \leftrightarrow \mathbf{8}_{\mathrm{s}} & \text{vacuum} \to |\mathbf{8}_{\mathrm{v}} + \mathbf{8}_{\mathrm{s}}, q\rangle \end{array}$$

Spectrum at level zero: massless

- Type IIA closed: $(\mathbf{8}_{\mathrm{v}} + \mathbf{8}_{\mathrm{s}}) \times (\mathbf{8}_{\mathrm{v}} + \mathbf{8}_{\mathrm{c}})$ (IIA supergravity)

$$\begin{aligned} \mathbf{8}_{\mathrm{v}} \times \mathbf{8}_{\mathrm{v}} + \mathbf{8}_{\mathrm{s}} \times \mathbf{8}_{\mathrm{c}} &= (\mathbf{35}_{\mathrm{v}} + \mathbf{28}_{\mathrm{v}} + \mathbf{1}) + (\mathbf{56}_{\mathrm{v}} + \mathbf{8}_{\mathrm{v}}), \\ \mathbf{8}_{\mathrm{v}} \times \mathbf{8}_{\mathrm{s}} + \mathbf{8}_{\mathrm{v}} \times \mathbf{8}_{\mathrm{c}} &= (\mathbf{56}_{\mathrm{s}} + \mathbf{8}_{\mathrm{c}}) + (\mathbf{56}_{\mathrm{c}} + \mathbf{8}_{\mathrm{s}}). \end{aligned}$$

- Type IIB closed: $(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})\times(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})$ (IIB supergravity)

$$\begin{aligned}\mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c}\times\mathbf{8}_\mathrm{c} &= (\mathbf{35}_\mathrm{v}+\mathbf{28}_\mathrm{v}+\mathbf{1})+(\mathbf{35}_\mathrm{c}+\mathbf{28}_\mathrm{v}+\mathbf{1}),\\ \mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{s}+\mathbf{8}_\mathrm{v}\times\mathbf{8}_\mathrm{s} &= (\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c})+(\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c}).\end{aligned}$$

- Type I closed: $(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})\times(\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c})$ mod $\mathbb{Z}_2$ (I supergravity)

$$(\mathbf{35}_\mathrm{v}+\mathbf{28}_\mathrm{v}+\mathbf{1})+(\mathbf{56}_\mathrm{s}+\mathbf{8}_\mathrm{c}).$$

- Type I open: $\mathbf{8}_\mathrm{v}+\mathbf{8}_\mathrm{c}$ (SYM).

10.3 Ramond–Neveu–Schwarz Superstring

There is an alternative formulation for the superstring: RNS. Manifest worldsheet rather than spacetime supersymmetry!

Action. Action in conformal gauge:

$$S\sim\int d^2\xi\,\eta_{\mu\nu}\left(\tfrac{1}{2}\partial_\mathrm{L}X^\mu\partial_\mathrm{R}X^\nu+i\Psi^\mu_\mathrm{L}\partial_\mathrm{R}\Psi^\nu_\mathrm{L}+i\Psi^\mu_\mathrm{R}\partial_\mathrm{L}\Psi^\nu_\mathrm{R}\right).$$

- action is supersymmetric.
- fermions are worldsheet spinors but spacetime vectors.

Bosons as before. Fermions can be periodic or anti-periodic.

Ramond Sector. $\Psi(\sigma+2\pi)=\Psi(\sigma)$ periodic.

- Fermion modes β_n as for bosons.
- Vacuum is a real 32-component fermionic spinor.
- $a=-\frac{1}{2}8\zeta(1)+\frac{1}{2}8\zeta(1)=0$.
- GSO projection: only chiral/anti-chiral states are physical!

Neveu–Schwarz Sector. $\Psi(\sigma+2\pi)=-\Psi(\sigma)$ anti-periodic.

- Half-integer modes for fermions: $\beta_{n+1/2}$.
- Vacuum is a bosonic scalar.
- $a=-\frac{1}{2}8\zeta(1)-\frac{1}{4}8\zeta(1)=\frac{1}{2}$.
- GSO projection: physical states require β^{2n+1}. No tachyon!

String Models. IIB/IIA strings for equal/opposite chiralities in L/R sectors.

Independent choice for left/right-movers in closed string. Four sectors: NS-NS, RN-S, NS-R, R-R. Independent vacua.

Superconformal Algebra. (Left) stress-energy tensor and conformal supercurrent:

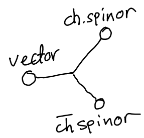
$$T_{\mathrm{L}} = \partial_{\mathrm{L}} X \cdot \partial_{\mathrm{L}} X + \tfrac{i}{2} \Psi_{\mathrm{L}} \cdot \partial_{\mathrm{L}} \Psi_{\mathrm{L}}, \qquad J_{\mathrm{L}} = \Psi_{\mathrm{L}} \cdot \partial_{\mathrm{L}} X$$

Superconformal algebra L_n, G_r ($2r$ is even/odd for R/NS):

$$\begin{aligned}
[L_m, L_n] &= (m-n) L_{m+n} + \tfrac{1}{8} c m (m^2 - 1) \delta_{m+n}, \\
[L_m, G_r] &= (\tfrac{1}{2} m - r) G_{m+r}, \\
\{G_r, G_s\} &= 2 L_{r+s} + \tfrac{1}{2} c (r^2 - \tfrac{1}{4}) \delta_{r+s}.
\end{aligned}$$

$c = D$ (conventional factor $\frac{3}{2}$ in c for super-Virasoro).

Comparison. GS and RNS approach yield the same results. In light cone gauge: related by $SO(8)$ triality



Compare features of both approaches:

	GS	RNS
fermions are spinors in	target space	worldsheet
worldsheet supersymmetry	(✓)	manifest
superconformal field theory	×	✓
target space supersymmetry	manifest	(✓)
supergravity couplings	all	some (NS-NS)
spacetime covariant	×	(✓)

Third approach exists: Pure spinors (Berkovits). Introduce auxiliary bosonic spinor λ satisfying $\lambda \gamma^\mu \lambda = 0$. Shares benefits of GS/RNS; covariant formulation.

10.4 Branes

Open superstrings couple to D-branes. Open string spectrum carries D-brane fluctuations.

- massless: $\mathcal{N} = 1$ Super-Yang–Mills reduced to $(d+1)$D.
- heavy string modes.
- sometimes: scalar tachyon.

Stable D*p*-Branes. D-branes can be stable or decay. Open string tachyon indicates D-brane instability.

- D-branes in bosonic string theory are instable.
- Dp-branes for IIB superstring are stable for p odd.
- Dp-branes for IIA superstring are stable for p even.

- T-duality maps between IIA and IIB.

Stability is related to supersymmetry. Boundary conditions break symmetry

- Lorentz: $SO(9,1) \to SO(d,1) \times SO(9-d)$.
- 16 supersymmetries preserved for p odd/even in IIB/IIA.
- no supersymmetries preserved for p even/odd in IIB/IIA.

Supersymmetry removes tachyon; stabilises strings.

Supergravity p-Branes. D-branes are non-perturbative objects. Not seen perturbatively due to large mass.

Stable Dp-branes have low-energy limit as supergravity solutions.

p-brane supported by $(p+1)$-form, gravity and dilaton.

- IIB/IIA have dilaton and two-form (NS-NS sector).
- IIB/IIA has forms of even/odd degree (R-R sector); relevant for stable Dp-branes.

Features:

- p-branes carry $(p+1)$-form charge. charge prevents p-branes from evaporating.
- charge density equals mass density.
- 16/32 supersymmetries preserved. 1/2 BPS condition.
- Non-renormalisation theorem for 1/2 BPS: p-branes same at weak/intermediate/strong coupling. BPS p-branes describe Dp-branes exactly.

Type-I Superstring. Consider open strings on D9-branes.

Gravity and gauge anomaly cancellation requires:

- gauge group of dimension 496.
- some special charge lattice property.

Two solutions: $SO(32)$ and $E_8 \times E_8$. Here: $SO(32)$. Breaks 1/2 supersymmetry: Type I.

- Sometimes considered independent type of superstring.
- Or: IIB, 16 D9 branes, space-filling orientifold-plane.

10.5 Heterotic Superstring

Two further superstring theories.

Almost no interaction between left and right movers. Exploit:

- left-movers as for superstring: 10D plus fermions.
- right-movers as for bosonic string: 26D (16 extra).

Heterotic string. 16 supersymmetries.

Anomaly cancellation requires gauge symmetry:

- HET-O: $SO(32)$ or

- HET-E: $E_8 \times E_8$.

Gauge group supported by 16 internal d.o.f..

HET-E interesting because E_8 contains potential GUT groups:

$$E_5 = SO(10), \qquad E_4 = SU(5), \qquad E_3 = SU(3) \times SU(2).$$

10.6 Dualities

Dualities relate seemingly different superstring theories.

- T-duality: time vs. space duality on worldsheet.
- S-duality: analog of electro-magnetic duality.

Dualities considered exact because of supersymmetry. Tests.

A Unique Theory. Dualities related various superstrings:

- T-duality: IIA $\leftrightarrow$ IIB; HET-E $\leftrightarrow$ HET-O
- S-duality: HET-O $\leftrightarrow$ Type I; IIB $\leftrightarrow$ IIB

Furthermore IIA and HET-E at strong coupling: 11D supergravity theory (with membrane).

Suspect underlying 11D theory called "M-theory". Superstring theories as various limits of M-theory.

Mirror Symmetry. Dualities applied to curved string backgrounds: Curved spacetimes with

- inequivalent metrics can have
- equivalent string physics.

E.g.: T-duality between large and small circles. Many examples for Calabi–Yau manifolds.

String/Gauge Duality. Some low-energy effective theories can become exact.

String physics at the location of a brane described exactly by corresponding YM theory.

Example: N coincident D3-branes in IIB string theory. Effective theory: $\mathcal{N} = 4$ Super-Yang–Mills theory in 4D.