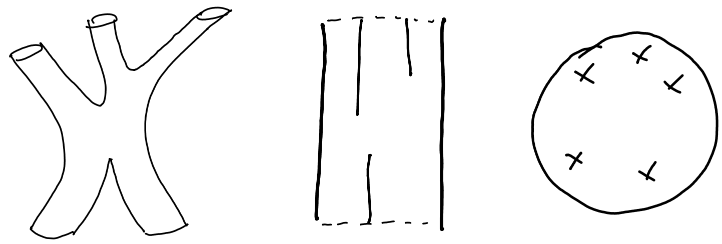
8 String Scattering

Compute a string scattering amplitude. Two methods:

- worldsheet junction(s). string cylinders with cuts. integration over junctions.
- vertex operators. integration over punctures locations.



8.1 Vertex Operators

State-operator map:

- Which operator creates a string?
- How to specify the momentum q?
- How to specify the string modes?

Solution is related to the operator $\mathcal{O}[q] = {:}\exp(iq_\mu X^\mu){:}$. Why?

- Momentum eigenstate: phase for translation $\exp(iq_\mu \epsilon^\mu)$.

Compute OPE with stress-energy T

$$T(z)\mathcal{O}[q](w,\bar{w}) = \frac{\frac{1}{4}\kappa^2 q^2 \mathcal{O}[q](w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}[q](w,\bar{w})}{z-w} + \ldots$$

Primary operator with weights $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$!

- non-trivial, non-integer weight,
- quantum effect $\sim \kappa^2$.

Consider two-point correlator

$$\big\langle \mathcal{O}_1[q_1]\mathcal{O}_2[q_2]\big\rangle \simeq |z_1 - z_2|^{\kappa^2(q_1\cdot q_2)}.$$

In fact, zero mode $X^\mu = x^\mu + \ldots$ contributes extra factor

$$\int d^D x \exp(iq_1 \cdot x + iq_2 \cdot x) \sim \delta^D(q_1 + q_2).$$

Hence compatible with primary of weight $(\frac{1}{4}\kappa^2 q^2, \frac{1}{4}\kappa^2 q^2)$

$$\big\langle \mathcal{O}_1[q_1]\mathcal{O}_2[q_2]\big\rangle \simeq \frac{\delta^D(q_1+q_2)}{|z_1 - z_2|^{\kappa^2 q_1^2}}\,.$$

Operator $\mathcal{O}[q](z,\bar{z})$ creates a string state at $(z,\bar{z})$. Worldsheet location unphysical, integrate:

$$V[q] = g_\mathrm{s} \int d^2z\, \mathcal{O}[q](z,\bar{z}).$$

Can only integrate weight $(1,1)$ primary operators. Hence:

- mass $M^2 = -q^2 = -4/\kappa^2$; string tachyon!
- intercept $a = \bar{a} = 1$ due to worldsheet integration.

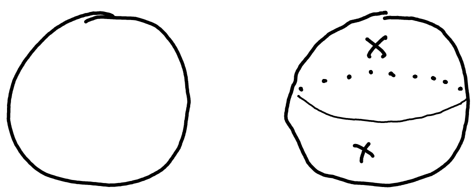
What about excited strings? Level-1 corresponds to

$$V^{\mu\nu}[q] = g_\mathrm{s} \int d^2z\, \partial X^\mu \bar{\partial} X^\nu \mathcal{O}[q].$$

- weight is $(1 + \frac{1}{4}\kappa^2 q^2, 1 + \frac{1}{4}\kappa^2 q^2) = (1,1)$ for massless q.
- primary condition removes unphysical polarisations.
- gauge d.o.f. are total derivatives.

Vertex operator picture:

- CFT vacuum is empty worldsheet (genus 0, no punctures).
- $\int d^2z\, \mathcal{O}[q](z,\bar{z})$ is string vacuum $|0;q\rangle$ (add puncture).
- $\int d^2z\, \ldots\, \mathcal{O}[q](z,\bar{z})$ are excited string states. Insertions of $\partial^n X^\mu$ correspond to string oscillators α^μ_n, insertions of $\bar{\partial}^n X^\mu$ correspond to $\bar{\alpha}^\mu_n$.



8.2 Veneziano Amplitude

Consider n-point amplitude (with $\mathcal{O}_k = \mathcal{O}[q_k](z_k,\bar{z}_k)$)

$$A_n \sim \frac{1}{g_\mathrm{s}^2}\langle V_1 \ldots V_n\rangle \sim g_\mathrm{s}^{n-2} \int d^{2n}z \langle \mathcal{O}_1 \ldots \mathcal{O}_n\rangle$$

- simplest to use tachyon vertex operators,
- can do others, but add complications (fields),
- computation & result qualitatively the same.

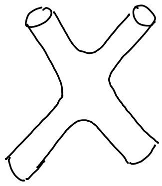
Perform Wick contractions and zero mode integration

$$\big\langle \mathcal{O}_1 \ldots \mathcal{O}_n \big\rangle \sim \delta^D(Q) \prod\nolimits_{j<k} |z_j - z_k|^{\kappa^2 q_j\cdot q_k}.$$

Integral invariant under Möbius transformations ($q_k^2 = 4/\kappa^2$). Map three punctures to fixed positions $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$. Remaining integral for $n = 4$ strings

$$A_4 \sim g_\mathrm{s}^2 \delta^D(Q) \int d^2z\, |z|^{\kappa^2 q_2\cdot q_4} |1 - z|^{\kappa^2 q_3\cdot q_4}$$

can be performed



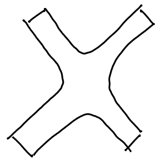
$$A_4 \sim g_s^2 \delta^D(Q) \frac{\Gamma(-1-\kappa^2 s/4)\,\Gamma(-1-\kappa^2 t/4)\,\Gamma(-1-\kappa^2 u/4)}{\Gamma(+2+\kappa^2 s/4)\,\Gamma(+2+\kappa^2 t/4)\,\Gamma(+2+\kappa^2 u/4)}\,.$$

Mandelstam invariants:

$$s = (q_1+q_2)^2, \quad t = (q_1+q_4)^2, \quad u = (q_1+q_3)^2,$$

with relation $s+t+u = -q_1^2 - q_2^2 - q_3^2 - q_4^2 = -16/\kappa^2$.

This is the Virasoro-Shapiro amplitude for closed strings. Corresponding amplitude for open strings

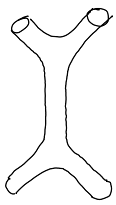


$$A_4 \sim g_s \frac{\Gamma(-1-\kappa^2 s)\,\Gamma(-1-\kappa^2 t)}{\Gamma(+2+\kappa^2 u)}$$

was proposed (not calculated) earlier by Veneziano. Considered birth of string theory (dual resonance model).

Amplitudes have many desirable features:

- Poles at $s,t,u = (N-1)4/\kappa^2$ or $s,t = (N-1)/\kappa^2$, virtual particles with string mass exchanged.



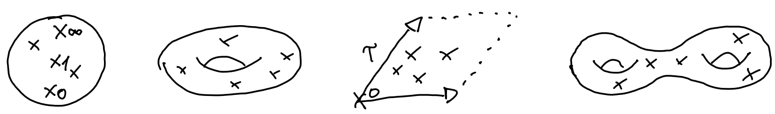
- Residues indicate spin $J = 2N$ or $J = N$. Regge trajectory!
- Soft behaviour at $s \to \infty$. Even for gravitons!
- Manifest crossing symmetry $s \leftrightarrow t \leftrightarrow u$ or $s \leftrightarrow t$. Amazing!

Not possible for QFT with finitely many particles.

8.3 String Loops

Result exact as far as α' is concerned. Free theory in α'!

However, worldsheet topology matters. String loop corrections for adding handles: higher genus. Power of g_s reflects Euler characteristic of worldsheet.



Tree Level. Worldsheet is sphere or disk with n punctures. Euler characteristic $-2+n$ or $-1+n/2$. 6 global conformal symmetries, integration over $n-3$ points.

One Loop. Worldsheet is torus with n punctures. Euler characteristic n. 2 moduli: integration over Teichmüller space. 2 shifts; integration over $n-1$ points. $2n$ integrations; result: elliptic & modular functions; feasible!

Two Loops. Worldsheet is 2-torus with n punctures. Euler characteristic $2+n$. 6 moduli, no shifts: $2(n+3)$ integrations. Hard, but can be done. No higher-loop results available.