4 String Quantisation

We have seen that the classical closed string is described by

- a bunch of harmonic oscillators $\alpha_n^{\mathrm{L/R}}$ for the string modes;
- a relativistic particle (x, p) describing the centre of mass.

Both systems coupled via Virasoro constraints.

We have two reasonable formulations:

- Covariant formulation with D oscillators α_n^μ per mode. Virasoro constraints $L_n^{\mathrm{R}} = L_n^{\mathrm{L}} = 0$ and conformal symmetry.
- Light cone formulation with $D-2$ oscillators $\vec{\alpha}_n$ per mode. No constraints, full Poincaré symmetry not manifest.

4.1 Canonical Quantisation

Derive commutation relations for variables x, p, α_n.

Recall action in conformal gauge

$$S = \frac{1}{2\pi\kappa^2}\int d^2\xi\, \tfrac{1}{2}\bigl(\dot{X}^2 - X'^2\bigr)$$

Substitute closed string mode expansion (with free time)

$$X^\mu = \kappa \sum\nolimits_n \beta_n^\mu(\tau)\, \exp(-in\sigma)$$

Obtain tower of HO's (β_0 free particle)

$$S = \frac{1}{2}\int d\tau \sum\nolimits_n \Bigl(\dot{\beta}_n \cdot \dot{\beta}_{-n} - n^2 \beta_n \cdot \beta_{-n}\Bigr)$$

Canonical momentum and canonical commutator:

$$\pi_n = \dot{\beta}_n, \qquad [\beta_m^\mu, \pi_n^\nu] = i\eta^{\mu\nu}\delta_{m+n}.$$

Match X with previous classical solution at $\tau = 0$

$$x^\mu = \kappa\beta_0^\mu, \qquad p^\mu = \frac{\pi_0^\mu}{\kappa}, \qquad \alpha_n^{\mathrm{L/R},\mu} = \frac{n\beta_{\mp n}^\mu}{i\sqrt{2}} + \frac{\pi_{\mp n}^\mu}{\sqrt{2}};$$

resulting commutators in original variables

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \qquad \bigl[\alpha_m^{\mathrm{L},\mu}, \alpha_n^{\mathrm{L},\nu}\bigr] = \bigl[\alpha_m^{\mathrm{R},\mu}, \alpha_n^{\mathrm{R},\nu}\bigr] = m\,\eta^{\mu\nu}\delta_{m+n}.$$

4.2 States

Compose space of states from free particle and oscillators.

- Momentum eigenstates for free particle $|q\rangle$.
- HO vacuum $|0\rangle$ and excitations for each mode/orientation.

Define string vacuum state $|0;q\rangle$

$$p^\mu|0;q\rangle = q^\mu|0;q\rangle, \qquad \alpha_n^{\mathrm{L/R},\mu}|0;q\rangle = 0 \quad \text{for } n>0.$$

Problem: negative norm states

$$|n,\mu;q\rangle := \alpha_{-n}^\mu|0;q\rangle, \qquad \big||n,\mu;q\rangle\big|^2 = \langle 0;q|\alpha_n^\mu\alpha_{-n}^\mu|0;q\rangle = n\eta^{\mu\mu}.$$

State not allowed by Virasoro constraints. General resolution: impose Virasoro constraints.

4.3 Light Cone Quantisation

Continue covariant quantisation later. Fix light cone gauge instead; only physical states.

Resulting commutators lead to positive definite states

$$\big[\alpha_m^{\mathrm{L},a},\alpha_n^{\mathrm{L},b}\big] = \big[\alpha_m^{\mathrm{R},a},\alpha_n^{\mathrm{R},b}\big] = m\,\delta^{ab}\delta_{m+n}.$$

Remember classical mass and residual constraint

$$M^2 = \frac{4}{\kappa^2}\sum_{m=1}^\infty \vec{\alpha}_{-m}^{\mathrm{L}}\cdot\vec{\alpha}_m^{\mathrm{L}} = \frac{4}{\kappa^2}\sum_{m=1}^\infty \vec{\alpha}_{-m}^{\mathrm{R}}\cdot\vec{\alpha}_m^{\mathrm{R}}.$$

Operator ordering matters! A priori free to choose. Assume normal ordering plus new constants $a^{\mathrm{L/R}}$:

$$M^2 = \frac{4}{\kappa^2}\left(\sum_{m=1}^\infty \vec{\alpha}_{-m}^{\mathrm{L}}\cdot\vec{\alpha}_m^{\mathrm{L}} - a^{\mathrm{L}}\right) = \frac{4}{\kappa^2}\left(\sum_{m=1}^\infty \vec{\alpha}_{-m}^{\mathrm{R}}\cdot\vec{\alpha}_m^{\mathrm{R}} - a^{\mathrm{R}}\right).$$

Combination measures string "level"

$$N := \sum_{m=1}^\infty \vec{\alpha}_{-m}\cdot\vec{\alpha}_m = \sum_{m=1}^\infty mN_m \quad \text{with } N_m := \frac{1}{m}\,\vec{\alpha}_{-m}\cdot\vec{\alpha}_m.$$

Mass and constraint in terms of string level

$$M^2 = \frac{4}{\kappa^2}(N^{\mathrm{L}} - a^{\mathrm{L}}) = \frac{4}{\kappa^2}(N^{\mathrm{R}} - a^{\mathrm{R}}).$$

4.4 String Spectrum

Mass depends on string level. Quantisation of string level $\longrightarrow$ quantisation of mass. Level matching: $N^{\mathrm{L}} - a^{\mathrm{L}} = N^{\mathrm{R}} - a^{\mathrm{R}}$.

Understand string states at each level; HO's.

Vacuum State. Define vacuum state $|0;q\rangle$

$$\vec{\alpha}_n^{\mathrm{L/R}}|0;q\rangle = 0 \quad \text{for } n>0.$$

Level zero: $N^{\mathrm{L}} = N^{\mathrm{R}} = 0$. Spin zero.

For physical state:

$$a^{\mathrm{R}} = a^{\mathrm{L}} = a, \qquad M^2 = -\frac{4a}{\kappa^2}\,.$$

So far so good: spin-0 particle with $M = 2\kappa^{-1}\sqrt{-a}$. $a \leq 0$?! Spatial extent: HO wave function $\sim \kappa$.

First Level. Lowest excited state has $N=1$. Level matching and $a^{\mathrm{L}} = a^{\mathrm{R}}$ implies $N^{\mathrm{L}} = N^{\mathrm{R}} = 1$. One excitation $\vec{\alpha}_{-1}$ each from left/right movers

$$|ab;q\rangle = \alpha_{-1}^{\mathrm{L},a}\alpha_{-1}^{\mathrm{R},b}|0;q\rangle.$$

$(D-2)^2$ states of mass $M = 2\kappa^{-1}\sqrt{1-a}$.

Spin under transverse rotations. Three combinations:

$$\begin{aligned}
|(ab);q\rangle &:= |ab;q\rangle + |ba;q\rangle - \frac{2\delta_{ab}}{D-2}|cc;q\rangle,\\
|[ab];q\rangle &:= |ab;q\rangle - |ba;q\rangle,\\
|1;q\rangle &:= |cc;q\rangle.
\end{aligned}$$

Transformation properties under $SO(D-2)$:

state	indices	Young tab.	"spin"
$\lvert(ab);q\rangle$	symmetric, traceless	$\square\square$	2
$\lvert[ab];q\rangle$	anti-symmetric	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	1
$\lvert 1;q\rangle$	singlet	$\bullet$	0

Stabiliser (little group) for massive particle is $SO(D-1)$. Can fit these $SO(D-2)$ reps. into $SO(D-1)$ reps.? No!

Only way out: massless particle; stabiliser $SO(D-2)$. set $a = a^{\mathrm{R}} = a^{\mathrm{L}} = 1$.

Three types of particles:

- $\boldsymbol{|(ab);q\rangle}$**:** massless spin-2 field. okay as free field.
  Weinberg–Witten: interactions are forbidden. except for gravitational interactions: graviton!
- $\boldsymbol{|[ab];q\rangle}$**:** massless 2-form field (Kalb–Ramond).
  $B_{\mu\nu}$ with 1-form gauge symmetry $\delta B_{\mu\nu} = \partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu$.
- $\boldsymbol{|1;q\rangle}$**:** massless scalar particle (dilaton).
  different from string vacuum $|0;q\rangle$.

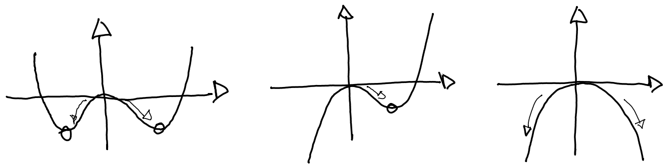
What we have learned:

- Interacting string theory includes gravity! κ is Planck scale.
- Graviton plus massless 2-form and scalar particles. Spatial extent $\sim \kappa$; practically point-like.
- $a = a^{\mathrm{R}} = a^{\mathrm{L}} = 1$.

Tachyon. Revisit string vacuum $|q, 0\rangle$: $M^2 = -4/\kappa^2 < 0$. Tachyon!

Problem? Not really, compare Goldstone/Higgs mechanism:

- Unstable vacuum at local maximum of a potential.
- Physical ground state at local minimum. No tachyon!
- Unclear if minimum exists. Where? What properties?



- Let us ignore. Indeed tachyon absent for superstrings!

Higher Levels. Levels zero and one work out. what about higher levels?

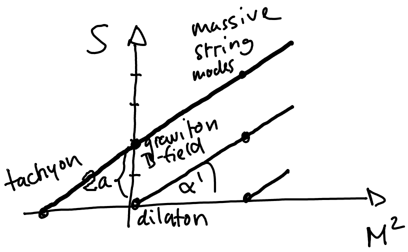
level	excitations	SO$(D-2)$	SO$(D-1)$
0	$\cdot$	$\bullet$	$\bullet$
1	α^a_{-1}	$\ydiagram{1}$	?
2	$\alpha^a_{-1}\alpha^b_{-1}$ α^a_{-2}	$\ydiagram{2} + \bullet$ $\ydiagram{1}$	$\ydiagram{2}$
3	$\alpha^a_{-1}\alpha^b_{-1}\alpha^c_{-1}$ $\alpha^a_{-1}\alpha^b_{-2}$ α^a_{-3}	$\ydiagram{3} + \ydiagram{1}$ $\ydiagram{2} + \ydiagram{1,1} + \bullet$ $\ydiagram{1}$	$\ydiagram{3}$ $\ydiagram{1,1}$
4	$\alpha^a_{-1}\alpha^b_{-1}\alpha^c_{-1}\alpha^d_{-1}$ $\alpha^a_{-1}\alpha^b_{-1}\alpha^c_{-2}$ $\alpha^a_{-2}\alpha^b_{-2}$ $\alpha^a_{-3}\alpha^b_{-1}$ α^a_{-4}	$\ydiagram{4} + \ydiagram{2} + \bullet$ $\ydiagram{3} + \ydiagram{2,1} + \ydiagram{1} + \ydiagram{1}$ $\ydiagram{2} + \bullet$ $\ydiagram{2} + \ydiagram{1,1} + \bullet$ $\ydiagram{1}$	$\ydiagram{4}$ $\ydiagram{2,1}$ $\ydiagram{2}$ $\bullet$
...	...	...	...

All higher levels combine into proper $SO(D-1)$ reps.. Level matching: need to square reps..

- String describes collection of infinitely many particle types.
- Various vibration modes might correspond to elementary particles. Including gravity.
- intrinsic particle extent κ. κ is Planck scale $\ll$ observed: Point-like particles!
- few light particles; all others at Planck mass; 1 tachyon?!
- very high excitations are long strings. mostly classical behaviour. $M \gg 1/\kappa$ superheavy.

Regge Trajectories. Maximum spin (symmetric indices) increases linearly with level $S = 2N$

$$M^2 = \frac{2S - 4a}{\kappa^2},$$



called leading Regge trajectory:

- $\alpha' = \kappa^2$ is Regge slope.
- $2a$ is Regge intercept, spin of massless particle.

Subleading trajectories for lower spins (indices anti-symmetric and in trace).

Qualitative similarity to hadron spectrum:

- Regge trajectories for hadronic resonances observed.
- $1/\kappa$ is QCD scale $\simeq 1$ GeV
- Intercept $a \approx -\frac{1}{2}$ rather than $a = 1$.
- another problem, see later.
- qualitative description of QCD flux tubes.

4.5 Anomalies

In light cone gauge we have broken manifest $SO(D-1, 1)$ Lorentz symmetry to a $SO(D-2)$ subgroup.

- Consequently spectrum of quantum strings organises manifestly into $SO(D-2)$ multiplets.
- Almost all multiplets fit into $SO(D-1)$ multiplets.
- Mass assignments fill Poincaré multiplets for $a^{\mathrm{L}} = a^{\mathrm{R}} = 1$.
- Poincaré symmetry broken unless $a^{\mathrm{L}} = a^{\mathrm{R}} = 1$.

Anomaly: Failure of classical symmetry in quantum theory.

Sometimes okay, not here, want strings to propagate on Minkowski background with intact Poincaré symmetry.

So far only counting, more severe problem in algebra. Commutator $[M^{-a}, M^{-b}]$ receives contributions from $[\alpha^-, \alpha^a]$:

$$[M^{-a}, M^{-b}] = \sum_{n=1}^{\infty} \left(\left(\frac{D-2}{24} - 1 \right) n + \left(a - \frac{D-2}{24} \right) \frac{1}{n} \right) \ldots$$

vanishes if and only if $D = 26$ and $a = 1$. String theory predicts twenty-six spacetime dimensions.

Shortcut derivation: reconsider nature of intercept a. a is sum of HO ground state energies $\frac{1}{2}\omega_n = \frac{1}{2}n$

$$a = -\sum_{n=1}^{\infty}(D-2)\tfrac{1}{2}\omega_n = -\tfrac{1}{2}(D-2)\sum_{n=1}^{\infty} n.$$

Sum divergent, black magic helps: ζ-function regularisation

$$\zeta(x) := \sum_{k=1}^{\infty}\frac{1}{k^x}, \quad \text{i.e.} \quad a = -\tfrac{1}{2}(D-2)\zeta(-1) = \frac{D-2}{24}\,.$$

Analytical continuation $\zeta(-1) = -\frac{1}{12}$ and $a = 1$ predicts $D = 26$! Murky derivation yields correct prediction.

4.6 Covariant Quantisation

In LC gauge Poincaré symmetry is subject to anomaly, but can also keep Poincaré manifest: Covariant quantisation. See how spectrum arises in covariant approach. Consider only L or R oscillators for simplicity.

Vacuum State. $|0;q\rangle$ defined as before. Satisfies

$$L_{n>0}|0;q\rangle = 0 \quad \text{and} \quad L_0|0;q\rangle = \frac{\kappa^2 q^2}{4}\,|0;q\rangle.$$

State not annihilated by negative Virasoro modes. Instead $\langle 0;q|L_{n<0} = 0$ hence $\langle 0;q|(L_n - \delta_n a)|0;q\rangle = 0$.

Impose Virasoro constraints for physical states $|\Psi\rangle$

$$L_{n>0}|\Psi\rangle = 0, \quad L_0|\Psi\rangle = a|\Psi\rangle, \qquad \langle\Psi|(L_n - \delta_n a)|\Psi\rangle = 0.$$

One Excitation. Generic ansatz for state

$$|\psi;q\rangle := \psi\cdot\alpha_{-1}|0;q\rangle.$$

Norm $\bar{\psi}\cdot\psi$ potentially negative. Virasoro constraint implies

$$L_1|\psi;q\rangle = \alpha_1\cdot\alpha_0|\psi;q\rangle = \frac{\kappa(\psi\cdot q)}{\sqrt{2}}\,|0;q\rangle = 0.$$

Furthermore $L_0 = a = 1$ implies $q^2 = 0$. Then $q\cdot\psi = 0$ removes negative norm state(s). Remains:

- $D-2$ states with positive norm.
- State with $\psi = q$ is null. Does not contribute to physics.

Two Excitations. Generic ansatz

$$|\phi, \psi; q\rangle := \phi_{\mu\nu}\alpha^\mu_{-1}\alpha^\nu_{-1}|0; q\rangle + \psi_\mu \alpha^\mu_{-2}|0; q\rangle.$$

Impose constraints $L_0, L_1, L_2 = 0$ to fix $q^2, \psi, \operatorname{tr}\phi$.

Remains: $\square\!\square$, $\square$, $\bullet$ of $SO(D-1)$.

- State $\square\!\square$ is positive definite.
- Ansatz for $\square$: $\phi_{\mu\nu} = q_\mu \rho_\nu + \rho_\mu q_\nu$ with $q \cdot \rho = 0$. Negative norm for $1 < a < 2$. Null state for $a = 1$!
- Ansatz for $\bullet$: $\phi_{\mu\nu} = q_\mu q_\nu + \eta_{\mu\nu}\sigma$. Negative norm for $D < 1$ or $D > 26$. Null state for $D = 26$!

Virasoro Algebra. Algebra of quantum charges L_n reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n}.$$

Latter term is central charge of Virasoro, $c = D$. We are interested in primary states

$$L_{n>0}|\Psi\rangle = 0, \quad L_0|\Psi\rangle = a|\Psi\rangle.$$

Can apply representation theory of Virasoro algebra $\Rightarrow$ CFT.

Proper treatment (BRST) includes ghosts. Classical conformal algebra when $D = 26$ and $a = 1$

$$[L_m, L_n] = (m-n)L_{m+n}.$$

Here: Conformal algebra is anomalous. Anomaly shifted to Lorentz algebra in light cone gauge.