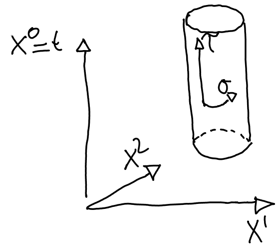
3 Classical Bosonic String

Mechanics of 1D extended object without inner structure.



- Worldsheet coordinates: $\xi^\alpha = (\tau, \sigma)$, time τ, space σ.
- Embedding coordinates $X^\mu(\xi)$.
- D-dimensional embedding Minkowski space, metric $\eta_{\mu\nu}$.

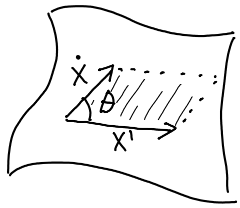
3.1 Nambu–Goto Action

Generalise insights from relativistic point particle:

- worldline $\longrightarrow$ worldsheet.
- action = proper time $\simeq$ "length" $\longrightarrow$ "area".

Area and Action. Wick rotation $t = iw$. Area of 2D euclidean surface:

$$\begin{aligned} dA &= d\tau\, d\sigma\, |X'|\, |\dot{X}|\, |\sin\theta| \\ &= d\tau\, d\sigma\, \sqrt{X'^2\, \dot{X}^2\, \sin^2\theta} \\ &= d\tau\, d\sigma\, \sqrt{X'^2\, \dot{X}^2 - (X' \cdot \dot{X})^2} \\ &= d^2\xi\, \sqrt{\det g}\,, \end{aligned}$$



induced worldsheet metric $g_{\alpha\beta} = \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ (pull back).

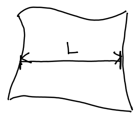
Wick rotate back; string action (Nambu–Goto)

$$S = -\frac{1}{2\pi\kappa^2} A = -\frac{1}{2\pi\kappa^2} \int d^2\xi\, \sqrt{-\det g}\,.$$

Symmetries of action:

- **Lorentz:** scalar products of Lorentz vectors X.
- **Poincaré:** X only through ∂X.
- **Worldsheet Diffeomorphisms:** density $d^2\xi\, \sqrt{-\det g}$ invariant under reparametrisations $\xi \mapsto \xi'(\xi)$.

Tension. What is parameter κ?



Fundamental string length scale $\longrightarrow$ quantum string.

Consider potential U: Time slice of action/area. Slice of length L: $U \sim L/\kappa^2$. Constant force $U' = 1/2\pi\kappa^2 = T$ is string tension. String not a spring or rubber band!

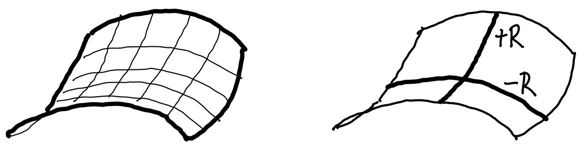
Equations of Motion. Vary action.

Use variation of determinant $\delta \det g = \det g\, g^{\alpha\beta} \delta g_{\alpha\beta}$.

$$\partial_\alpha \big(\sqrt{-\det g}\, g^{\alpha\beta} \partial_\beta X^\mu \big) = 0.$$

Highly non-linear equations (g contains X)! How to solve? How to deal with?

Geometrically: stationary action; minimal area surface. Static soap films! Mean curvature zero. Saddle point everywhere, equal/opposite sectional curvatures.



3.2 Polyakov Action

Complication from non-linear e.o.m.. As for point particle, there is a polynomial action with additional dynamical worldsheet metric $g_{\alpha\beta}$:

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi \sqrt{-\det g}\, \tfrac{1}{2} g^{\alpha\beta} (\partial_\alpha X) \cdot (\partial_\beta X).$$

E.o.m. for X as above; e.o.m. for worldsheet metric g:

$$(\partial_\alpha X) \cdot (\partial_\beta X) = \tfrac{1}{2} g_{\alpha\beta}\, g^{\gamma\delta} (\partial_\gamma X) \cdot (\partial_\delta X).$$

Solution fixes induced metric (up to local scale f)

$$g_{\alpha\beta} = f(\xi)\, (\partial_\alpha X) \cdot (\partial_\beta X).$$

Arbitrary scale $f(\xi)$ cancels in action and all e.o.m.. New redundancy: **Weyl invariance** $g_{\alpha\beta}(\xi) \mapsto f(\xi) g_{\alpha\beta}(\xi)$.

3.3 Conformal Gauge

E.o.m. for X linear, coupling to g makes non-linear.

Use gauge freedom: demand conformally flat metric.

$$g_{\alpha\beta}(\xi) = f(\xi) \eta_{\alpha\beta}.$$

Amounts to two equations $g_{\tau\sigma}(\xi) = 0$, $g_{\tau\tau}(\xi) = -g_{\sigma\sigma}(\xi)$.

- Fixes almost all diffeomorphisms.
- Conformal transformations remain. Diffeomorphisms preserving metric up to scale.
- May further set $f = 1$ by means of fixing Weyl.

Action describes D free massless scalar particles in 2D

$$S = -\frac{1}{2\pi\kappa^2} \int d^2\xi \tfrac{1}{2} \eta^{\alpha\beta} (\partial_\alpha X) \cdot (\partial_\beta X).$$

E.o.m. simply $\partial^2 X^\mu = 0$ (harmonic wave equation)

$$\ddot{X} = X''.$$

Do not forget e.o.m. for worldsheet metric

$$T_{\alpha\beta} := (\partial_\alpha X) \cdot (\partial_\beta X) - \tfrac{1}{2} \eta_{\alpha\beta} \eta^{\gamma\delta} (\partial_\gamma X) \cdot (\partial_\delta X) = 0.$$

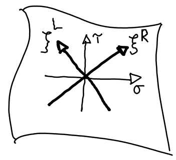
$T_{\alpha\beta}$ is energy-momentum tensor for D scalars. Trace absent $T^\alpha_\alpha = 0$ by construction (Weyl/conformal). Two remaining e.o.m. become **Virasoro constraints**

$$\dot{X} \cdot X' = 0, \qquad \dot{X} \cdot \dot{X} + X' \cdot X' = 0.$$

Forbids longitudinal waves along the worldsheet, no structure!

Conservation of $T_{\alpha\beta}$: impose constraints only on time slice.

3.4 Solution on the Light Cone



Solve harmonic wave equation. Light cone coordinates $\xi^{\mathrm{L/R}}$ useful:

$$\xi^{\mathrm{L/R}} = \tau \mp \sigma, \quad \partial_{\mathrm{L/R}} = \tfrac{1}{2}(\partial_\tau \mp \partial_\sigma),$$

new worldsheet metric

$$d^2 s = -d\tau^2 + d\sigma^2 = -d\xi^{\mathrm{L}} d\xi^{\mathrm{R}}.$$

Now e.o.m. and Virasoro constraints read

$$\partial_{\mathrm{L}} \partial_{\mathrm{R}} X^\mu = 0, \qquad (\partial_{\mathrm{L/R}} X)^2 = 0.$$

First equation solved by simple separation of variables

$$X^\mu(\xi^{\mathrm{L}}, \xi^{\mathrm{R}}) = X^\mu_{\mathrm{L}}(\xi^{\mathrm{L}}) + X^\mu_{\mathrm{R}}(\xi^{\mathrm{R}}).$$

D left-movers X_{L} plus D right-movers X_{R}. Virasoro constraints $(\partial X_{\mathrm{R,L}})^2 = 0$ remove 1 left/right-mover. Two reparametrisations left:

- **conformal transformations**

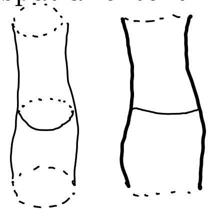
$$\xi^{\mathrm{R}} \mapsto \xi'^{\mathrm{R}}(\xi^{\mathrm{R}}), \qquad \xi^{\mathrm{L}} \mapsto \xi'^{\mathrm{L}}(\xi^{\mathrm{L}}).$$

  2D case special: infinitely many transformations. removes another 1 left/right-mover.
- **constant shift** between X^μ_{L} and X^μ_{R}.

$(D-2)$ left/right-movers remain (transverse).

3.5 Closed String Modes

So far worldsheet infinitely extended in space and time; want finite spatial extent.



- **Closed String:** circular topology.
  Identify $\sigma \equiv \sigma + 2\pi$ (other choices possible).
- **Open String:** interval topology.
  Boundary conditions at $\sigma = 0, \pi$ (later).

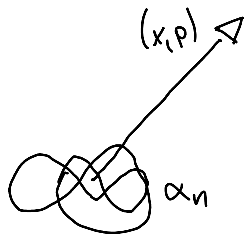
Periodic function $X \longrightarrow$ Fourier decomposition

$$X^\mu_{\mathrm{L/R}} = \tfrac{1}{2}x^\mu + \tfrac{1}{2}\kappa^2 p^\mu \xi^{\mathrm{L/R}} + \sum_{n\neq 0} \frac{i\kappa}{\sqrt{2}n}\,\alpha_n^{\mathrm{L/R},\mu} \exp(-in\xi^{\mathrm{L/R}}).$$

- Coefficients $i\kappa/\sqrt{2}n$ chosen for later convenience.
- Linear dependence in $\xi^{\mathrm{L/R}}$ okay: $X^\mu = x^\mu + \kappa^2 p^\mu \tau + \ldots$.
- Reality of X: complex conjugate $\alpha_{-n} = (\alpha_n)^*$.

Two kinds of parameters for solutions

- Centre of mass motion x, p (conjugate variables $\to \kappa^2$).
- String modes $\alpha_n^{\mathrm{L/R},\mu}$ (left/right movers).



Consider Virasoro constraints, substitute modes:

$$(\partial_{\mathrm{L,R}} X^\mu_{\mathrm{L/R}})^2 = \kappa^2 \sum\nolimits_n L_n^{\mathrm{L/R}} \exp(-in\xi^{\mathrm{L/R}}) \stackrel{!}{=} 0$$

with Virasoro modes (drop L/R index)

$$L_n := \tfrac{1}{2}\sum\nolimits_m \alpha_{n-m}\cdot\alpha_m \stackrel{!}{=} 0.$$

Note that $\alpha_0^{\mathrm{L}} = \alpha_0^{\mathrm{R}} = \kappa p/\sqrt{2}$. $L_0 = 0$ constraint fixes string mass

$$p^2 = -M^2, \qquad M^2 = \frac{4}{\kappa^2}\sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m$$

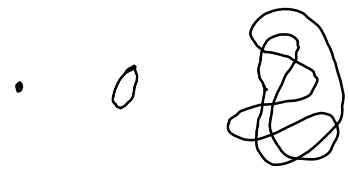
All Virasoro constraints $L_n = 0$ conserved by e.o.m.

$$\dot{L}_n = inL_n.$$

Need to impose on initial data only.

String mass depends on mode amplitudes.

- No modes excited: massless point particle.
- Few/small excitations: light tiny particle.
- Many/large excitations: big massive object.



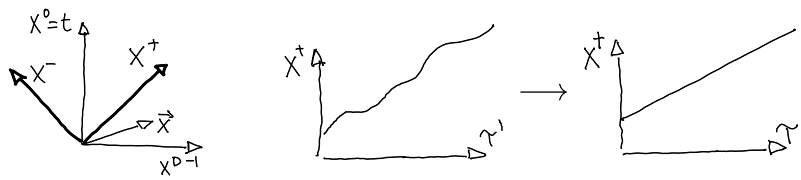
Note: time-like modes α^0 contribute negative M^2. Tachyons excluded by Virasoro constraints.

Virasoro constraints non-linear, complicated.

3.6 Light Cone Gauge

Use conformal symmetry to solve Virasoro constraints.

Gauge Fixing. Introduce as well light cone coordinates for spacetime $X^\pm = X^0 \pm X^{D-1}$. $\vec{X}$ denotes components $1 \ldots (D-2)$. Gauge fix!



$$X^+_{\mathrm{L/R}} = x^+_{\mathrm{L/R}} + \tfrac{1}{2}\kappa^2 p^+_{\mathrm{L/R}} \xi^{\mathrm{L/R}}.$$

Virasoro constraint $(\partial \vec{X}_{\mathrm{L/R}})^2 - \kappa^2 p^+_{\mathrm{L/R}} \partial X^-_{\mathrm{L/R}} = 0$ solved by

$$X^-_{\mathrm{L/R}}(\xi) = \frac{1}{\kappa^2 p^+_{\mathrm{L/R}}} \int^\xi d\xi' \big(\partial \vec{X}_{\mathrm{L/R}}(\xi')\big)^2.$$

Solution: $2(D-2)$ arbitrary functions $\vec{X}_{\mathrm{L/R}}(\xi^{\mathrm{L/R}})$.

Periodicity. All functions $\vec{X}(\xi)$ periodic! Furthermore: periodicity of X^+ and X^- requires

$$p^+_{\mathrm{L}} = p^+_{\mathrm{R}} = p^+, \qquad \int_0^{2\pi} d\xi \big((\partial \vec{X}_{\mathrm{R}})^2 - (\partial \vec{X}_{\mathrm{L}})^2\big) = 0.$$

Residual gauge freedom: Constant shift $\Delta X^\mu_{\mathrm{R}}(\xi) = -\Delta X^\mu_{\mathrm{L}}(\xi)$. Corresponds to above residual constraints.

String Modes. Impose gauge fixing on modes ($n \neq 0$)

$$\alpha_n^+ = 0, \qquad \alpha_n^- = \frac{1}{\alpha_0^+} \sum\nolimits_m \vec{\alpha}_{n-m} \cdot \vec{\alpha}_m.$$

Periodicity requires $\vec{\alpha}_0^{\mathrm{L}} = \vec{\alpha}_0^{\mathrm{R}}$, $\alpha_0^{\mathrm{R},+} = \alpha_0^{\mathrm{L},+}$ and for α_0^-

$$\sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\mathrm{L}} \cdot \vec{\alpha}_m^{\mathrm{L}} = \sum_{m=1}^{\infty} \vec{\alpha}_{-m}^{\mathrm{R}} \cdot \vec{\alpha}_m^{\mathrm{R}}$$

Resulting mass manifestly positive:

$$M^2 = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} \vec{\alpha}_{-m} \cdot \vec{\alpha}_m = \frac{4}{\kappa^2} \sum_{m=1}^{\infty} |\vec{\alpha}_m|^2$$

Benefits: positivity, almost all constraints gone. Drawback: Lorentz symmetry not manifest.