Exercise 9.1 Ideal Quantum Gases in a Harmonic Trap

In this exercise we will discuss the difference between bosons and fermions. In order to do that, we compare bosonic and fermionic quantum corrections to a spinless ideal gas confined in a three-dimensional harmonic potential.[1] The energy states of the gas are given by

$$E_{\mathbf{a}} = \hbar\omega(a_x + a_y + a_z) \ , \tag{1}$$

where, as usually, we neglect the zero point energy of $E_0 = 3\,\hbar\omega/2$. Here, $\mathbf{a} = (a_x, a_y, a_z)$ denotes the occupation number of oscillator modes of the state $E_{\mathbf{a}}$ ($a_i = 0, 1, 2, \ldots$).

a) Consider the high-temperature, low-density limit ($z \ll 1$). Derive the grand canonical partition function $\mathcal{Z}_{b,f}$ of this system and compute the grand potential $\Omega_{b,f}$ for bosons and fermions. Show that

$$\Omega_f \propto f_4(z) \ , \qquad \Omega_b \propto g_4(z) \ , \tag{2}$$

where the functions $f_s(z)$ and $g_s(z)$ are defined as

$$f_s(z) = -\sum_{l=1}^{\infty} (-1)^l \frac{z^l}{l^s} \ , \qquad g_s(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^s} \ . \tag{3}$$

b) Derive the internal energy, U, the specific heat, C_N and the particle number, N, of both the bosonic and the fermionic systems.

In order to relate U and C_N to N, introduce an expansion in a small parameter which depends on the particle number instead of the chemical potential.

Define an effective volume, $\mathcal{V}_{\text{eff}}$, in terms of the average square displacement of the harmonic oscillator, $\langle r^2 \rangle$, as $\mathcal{V}_{\text{eff}} = 4\pi/3\,\langle r^2\rangle^{3/2}$. Give an interpretation for this quantity. Define and compute the thermal expansion coefficient, α, using $\mathcal{V}_{\text{eff}}$.

c) Interpret your results of part b) by comparing them with corresponding results for the classical Boltzmann gas. How do quantum corrections influence bosonic and fermionic systems?

d) Find the critical temperature, T_c, at which Bose-Einstein condensation sets in. How can we conciliate this condensation with the high-temperature, low-density limit?

Exercise 9.2 Bose-Einstein Condensation

a) In Section 4.5 of the lecture notes we derived an expression for the specific heat, C_V, of the spinless Bose gas, above and below the critical temperature, T_c (eq. 4.81). In fig. 4.4 of the same section, we see that C_V does not diverge at T_c, but it has a cusp there, suggesting that a T-derivative of C_V does diverge.

[1] For results of the classical ideal gas in a harmonic trap see Section 2.4.3 of the lecture notes

Evaluate

$$\Delta = \lim_{T \to T_c^+} \partial_T \, C_V(T) - \lim_{T \to T_c^-} \partial_T \, C_V(T) \neq 0 \,, \tag{4}$$

to show that $\partial_T \, C_V(T)$ is not continuous at T_c, and so $\partial_T^2 \, C_V(T)$ diverges there.

b) As we saw in a), in the vicinity of a phase-transition several thermodynamic quantities may show non-analytic behavior. The way in which these quantities diverge gives us useful information about the transition itself. Usually one can find a power-law behavior, $X(T) \propto (T - T_c)^\gamma$, for some quantity X, near the transition temperature, T_c. Here, γ is often called a *critical exponent*.

   Show that the compressibility of the Bose gas, κ_T, shows power-law behavior near T_c, and find the corresponding critical exponent.

   Hint: At $T = T_c$, we have $z = 1$, so we can expand κ_T in $\nu := \ln z$ around $\nu = 0$.

c) So far we considered a spinless Bose gas; however, fermions and bosons may have spin, and magnetic properties become important in those systems.

   Adapt the calculation of the spin-susceptibility in ex. 8.2 to the case of bosons with spin, and show that it diverges at $T = T_c$, in the limit $B \to 0$.

   What kind of magnetism do you expect the bosonic system exhibit for $T < T_c$? Try to give simple arguments for your conclusions.

Office Hours: Monday, November 21, 8-10 am (Tama Ma, HIT K 31.3)