# Sheet 7

Deadline: 21 November 2011

**Exercise 1** [*Quantisation of scalar field from transformation property* ]:
In this exercise we want to deduce the canonical commutation relations for the free massive scalar field from the requirement that the stress energy tensor induces the appropriate space-time transformations. Make an ansatz for the scalar field with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\,m^2\,\phi^2 \tag{1}$$

as

$$\phi(t,\mathbf{x}) = \int d\tilde{k}\ \left[a(k)\,e^{-ik\cdot x} + a^\dagger(k)\,e^{ik\cdot x}\right]\ , \tag{2}$$

and recall that the stress energy tensor is defined as

$$T^\mu{}_\nu \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\nu\phi_i - \delta^\mu{}_\nu\mathcal{L}\ . \tag{3}$$

Express the momentum operators

$$P^i = \int d^3\mathbf{x}\,T^{0i} \tag{4}$$

in terms of the modes $a(k)$ and $a^\dagger(k)$, and determine their commutation relations from the requirement that

$$[P^i,\phi(t,\mathbf{x})] = -i\partial^i\phi(t,\mathbf{x})\ . \tag{5}$$

**Exercise 2** [*Feynman propagator for Dirac fields* ]:
Time ordering for fermionic fields is defined by

$$\mathcal{T}\left(\psi_\xi(x)\,\bar\psi_\eta(y)\right) = \theta(x^0-y^0)\,\psi_\xi(x)\,\bar\psi_\eta(y) - \theta(y^0-x^0)\,\bar\psi_\eta(y)\,\psi_\xi(x)\ . \tag{6}$$

The fermionic Dirac fields can be written in terms of creation and annihilation operators as

$$\psi(x) = \sum_{\alpha=1,2}\int d\tilde{k}\,\left(b_\alpha(k)u^{(\alpha)}(k)e^{-ik\cdot x} + d_\alpha^\dagger(k)v^{(\alpha)}(k)e^{ik\cdot x}\right)\ , \tag{7}$$

with anticommutation relations

$$\{b_\alpha(k),b_\beta^\dagger(k')\} = \{d_\alpha(k),d_\beta^\dagger(k')\} = (2\pi)^3 2k^0\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta_{\alpha\beta}\ . \tag{8}$$

Calculate the time ordered expectation value of two Dirac fields and show that it equals

$$\begin{aligned} iS_F(x-y)_{\xi\eta} &\equiv \langle 0|\,\mathcal{T}\left(\psi_\xi(x)\,\bar\psi_\eta(y)\right)\,|0\rangle \\ &= i\int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x-y)}\,\left(\frac{\not{k}+m}{k^2-m^2+i\epsilon}\right)_{\xi\eta}\ . \end{aligned} \tag{9}$$

Check that it can be expressed in terms of the Feynman propagator of the scalar field theory as

$$S_F(x) = -\,(i\not{\partial}+m)\,G_F(x) = (i\not{\partial}+m)\,\int\frac{d^4k}{(2\pi)^4}\,\frac{e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}\ . \tag{10}$$

**Exercise 3** [*Spinor products* ]:
Let $k_0^\mu, k_1^\mu$ be fixed 4-vectors satisfying $k_0^2 = 0$, $k_1^2 = -1$, $k_0 \cdot k_1 = 0$. Define basic spinors in the following way: Let $u_{L0}$ be the left-handed spinor for a fermion with momentum $k_0$, and define $u_{R0} = \not{k}_1 u_{L0}$. Then, for any $p$ such that $p$ is lightlike ($p^2 = 0$), define

$$u_L(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_{R0} \qquad \text{and} \qquad u_R(p) = \frac{1}{\sqrt{2p \cdot k_0}} \not{p} u_{L0}. \tag{11}$$

This set of conventions defines the phases of spinors unambiguously (except when $p$ is parallel to $k_0$).

(i) Show that $\not{k}_0 u_{R0} = 0$. Show that, for any lightlike $p$, $\not{p} u_L(p) = \not{p} u_R(p) = 0$.

(ii) For the choices $k_0 = (E, 0, 0, -E)$, $k_1 = (0, 1, 0, 0)$, construct $u_{L0}$, $u_{R0}$, $u_L(p)$, and $u_R(p)$ explicity.

(iii) Define the *spinor products* $s(p_1, p_2)$ and $t(p_1, p_2)$, for $p_1$, $p_2$ lightlike, by

$$s(p_1, p_2) = \bar{u}_R(p_1) u_L(p_2) \ , \qquad t(p_1, p_2) = \bar{u}_L(p_1) u_R(p_2) \ . \tag{12}$$

Using the explicit forms for the $u_\lambda$ given in part (ii), compute the spinor products explicitly and show that $t(p_1, p_2) = (s(p_1, p_2))^*$ and $s(p_1, p_2) = -s(p_2, p_1)$. In addition, show that

$$|s(p_1, p_2)|^2 = 2 p_1 \cdot p_2 \ . \tag{13}$$

Thus the spinor products are the square roots of 4-vector dot products.

*Hint:* In the chiral basis,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \ , \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

the normalised Dirac spinor can be written as

$$u^\alpha(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi^\alpha \\ \sqrt{p \cdot \bar{\sigma}}\, \xi^\alpha \end{pmatrix} \ ,$$

where $\xi$ is a two-component spinor normalised to unity.