Sheet 5

Deadline: 07 November 2011

Exercise 1 [*Wick's Theorem*]:

The aim of this exercise is to prove by induction on the number of fields Wick's Theorem

$$\mathcal{T}\Big(\phi(x_1)\cdots\phi(x_n)\Big) = \mathcal{N}\Big(\phi(x_1)\cdots\phi(x_n) + \text{all possible contractions}\Big)\,. \tag{1}$$

Here the normal ordering only applies to the non-contracted fields, and time ordering is defined by

$$\mathcal{T}\left(\phi(x_1)\cdots\phi(x_n)\right) = \phi(x_{\pi(1)})\cdots\phi(x_{\pi(n)})\,, \qquad x^0_{\pi(1)} > x^0_{\pi(2)} > \cdots > x^0_{\pi(n)}\ , \tag{2}$$

while the normal ordering moves the annihilation operators to the right of creation operators

$$\mathcal{N}\Big(\phi(x_1)\,\phi(x_2)\Big) = \phi(x_1)_+\,\phi(x_2)_+ + \phi(x_1)_+\,\phi(x_2)_- + \phi(x_2)_+\,\phi(x_1)_- + \phi(x_1)_-\,\phi(x_2)_-\ , \tag{3}$$

where $\phi = \phi_+ + \phi_-$ with ϕ_+ containing the creation generators and ϕ_- the annihilation generators. Furthermore we denote by the contraction of two operators the quantity

$$\phi(\overline{x_1)\phi(x_2}) = \theta(x^0_1 - x^0_2)\,[\phi_-(x_1), \phi_+(x_2)] + \theta(x^0_2 - x^0_1)\,[\phi_-(x_2), \phi_+(x_1)]\ . \tag{4}$$

(i) Prove that the contraction is a complex number (rather than an operator), and show that it agrees with $-iG_{\rm F}(x_1 - x_2)$ that was calculated on Sheet 1, Exercise 2 (v).

(ii) Show first that (1) holds for the case of two fields.

(iii) Explain why one may assume, without loss of generality, that the fields in (1) are already time-ordered.

(iv) Then prove the formula by induction on the number of fields, i.e. assume that the formula is true for m fields, and deduce it for $m + 1$ fields.

Exercise 2 [*Explicit solutions of the Wightman-propagator*]:

As shown in the lectures the commutation relation of two scalar fields does not vanish at different times, and is given by

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \mathrm{d}\tilde{k}\,\left[e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right] \qquad (5)\\ &= i\Big(\Delta_+(x - y) - \Delta_+(y - x)\Big) \\ i\Delta_+(z) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{p}}{2\omega_{\mathbf{p}}} e^{-ip\cdot z}\ , \qquad (6)\end{aligned}$$

where $\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2\omega_{\mathbf{k}}}$, and $\Delta_+(x)$ is the so called *Wightman* propagator. In special cases this propagator can be calculated explicitely.

(i) Show that for spacelike x the solution of $\Delta_+(x)$ is a K-type Bessel function

$$\Delta_+(x) = \frac{m}{4\pi^2\sqrt{-x^2}} K_1(m\sqrt{-x^2}) \ , \tag{7}$$

and hence deduce that the *Feynman* propagator $G_\mathrm{F}(x)$ drops off exponetially for large $|\mathbf{x}|$.

Hint: Use that

$$K_\nu(z) = \frac{\Gamma(\nu+1/2)(2z)^\nu}{\sqrt{\pi}} \int_0^\infty \mathrm{d}t \, \frac{\cos(t)}{(t^2+z^2)^{\nu+1/2}} \ , \tag{8}$$

where $\Gamma(w)$ is the Gamma function with particular value $\Gamma(3/2) = 1/2\sqrt{\pi}$. Then note that the relation between the Feynman and the Wightman propagator is

$$G_\mathrm{F}(z) = \theta(z^0)\Delta_+(z) + \theta(-z^0)\Delta_+(-z) \ . \tag{9}$$

(ii) For the case of $m = 0$ compute both $\Delta_+(x)$ and $G_\mathrm{F}(x)$.

Hint: Show and use that

$$\int_{S^2} e^{i|\mathbf{p}|\mathbf{w}\cdot\mathbf{x}} \mathrm{d}w = \frac{4\pi\sin(|\mathbf{p}||\mathbf{x}|)}{|\mathbf{p}||\mathbf{x}|}, \quad |\mathbf{w}| = 1 \tag{10}$$

$$-i\int_0^\infty \mathrm{d}u \, e^{isu} = P\left(\frac{1}{s}\right) - i\pi\delta(s) \ , \tag{11}$$

where P is the principal value.