Sheet 4

Deadline: 31 October 2011

Exercise 1 [*Lorentz generators*]:
It has been shown in the lectures that the Noether charges corresponding to the Lorentz transformations are given in the free scalar theory by

$$M^{\mu\nu} = \int d^3\mathbf{x}\, \left(x^\mu T^{0\nu} - x^\nu T^{0\mu}\right) , \tag{1}$$

where $T^{\mu\nu}$ is the stress energy tensor

$$T^{\mu\nu} = \partial^\mu\phi\,\partial^\nu\phi - \frac{1}{2}g^{\mu\nu}\,(\partial_\rho\phi)^2 + \frac{1}{2}g^{\mu\nu}\,m^2\phi^2 \ . \tag{2}$$

Show that the charges can be written using creation and annihilation operators $a^\dagger(k)$ and $a(k)$ as

$$\begin{aligned} M_{0j} &= i\int d\tilde{k}\, a^\dagger(k)\left(\omega_{\mathbf{k}}\frac{\partial}{\partial k^j}\right) a(k) \\ M_{jl} &= i\int d\tilde{k}\, a^\dagger(k)\left(k_j\frac{\partial}{\partial k^l} - k_l\frac{\partial}{\partial k^j}\right) a(k) \ , \end{aligned} \tag{3}$$

where $d\tilde{k} = d^3\mathbf{k}\,\frac{1}{(2\pi)^3 2\omega_{\mathbf{k}}}$.

Exercise 2 [*Poincare algebra*]:
By using the representation of the Poincaré transformations as differential operators

$$P^\mu = -i\partial^\mu \ , \qquad M^{\mu\nu} = -i(x^\mu\partial^\nu - x^\nu\partial^\mu) \tag{4}$$

deduce the commutation relations of the Poincaré algebra

$$\begin{aligned} [P^\mu, P^\nu] &= 0 \\ [M^{\mu\nu}, P^\lambda] &= i(g^{\mu\lambda}\,P^\nu - g^{\nu\lambda}\,P^\mu) \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i(g^{\mu\rho}\,M^{\nu\sigma} - g^{\nu\rho}\,M^{\mu\sigma} - g^{\mu\sigma}\,M^{\nu\rho} + g^{\nu\sigma}\,M^{\mu\rho}) \ . \end{aligned} \tag{5}$$

Exercise 3 [*Representation of Poincare algebra*]:
Recall that the momentum operator of the free scalar theory can be written as

$$P^\mu = \int d\tilde{k}\, k^\mu\, a^\dagger(k)\, a(k) \ . \tag{6}$$

Using this expression and the commutation relations $[a(k), a^\dagger(k')] = (2\pi)^3\, 2\omega_{\mathbf{k}}\, \delta^{(3)}(\mathbf{k}-\mathbf{k}')$ show that the generators (3) from Exercise 1 satisfy the commutation relations of the Poincaré algebra (5).

Exercise 4 [*Charges as generators*]:
The fact that the conserved charges associated to a transformation are also their generators is a general fact in any quantum field theory. Let us show this statement for the case of a scalar field theory. Recall from the lectures that if the Langrangian density $\mathcal{L}[\phi]$ is invariant under an infinitesimal transformation $\phi \to \phi + \Delta\phi$, then the corresponding conserved charge Q is given by

$$Q(t) = \int d^3x \, j^0(\vec{x}, t) \; , \qquad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \; .$$

Since $\partial_0 Q = 0$, we can write $Q(t) \equiv Q$. Prove, using the canonical quantisation relation, that Q is indeed the generator of the transformation $\phi \to \phi + \Delta\phi$, i.e. that we have

$$[Q, \phi(x)] = i\Delta\phi \; .$$