Sheet 3

Deadline: 24 October 2011

Exercise 1 [*Lorentz group*]: Recall the commutation relations of the Lorentz group from the last example sheet

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(g^{\nu\rho}\,\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\,\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\,\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\,\mathcal{J}^{\nu\rho}\right) \ . \tag{1}$$

(i) Define the generators of rotations and boosts as

$$L^i = \frac{1}{2}\epsilon^{ijk}\mathcal{J}^{jk} \ , \qquad K^i = \mathcal{J}^{0i} \ , \tag{2}$$

where $i, j, k = 1, 2, 3$. Determine the commutation relations of L^i and K^j among each other, and conclude that the combinations

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \ , \qquad \mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K}) \tag{3}$$

commute with one another and satisfy the commutation relations of angular momentum. The finite-dimensional representations of the (covering group of the) Lorentz group can hence be described in terms of two half-integers (j_1, j_2), specifying the representations of the $\mathbf{J}_+$ and $\mathbf{J}_-$ generators, respectively.

(ii) As shown in Ex 3 of Sheet 2, the operators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{4}$$

define a representation of the Lorentz algebra. Show that in the Weyl representation of the γ-matrices, we have

$$\mathbf{J}_+ = \frac{1}{2}\begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{J}_- = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}. \tag{5}$$

Hence conclude that that the Dirac spinor transforms in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the quantum mechanical Lorentz group.

(iii) Let us write the Dirac spinor in terms of two Weyl-spinors, ψ_L and ψ_R, where ψ_L transforms in the $(\frac{1}{2}, 0)$ representation, while ψ_R transforms in $(0, \frac{1}{2})$. Define $\psi' = \psi_L^t\,\epsilon$, where

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \tag{6}$$

and show that ψ' tranforms trivially under $\mathbf{J}_-$, while under $\mathbf{J}_+$ it transforms as

$$\psi' \mapsto -\frac{1}{2}\,\psi'\sigma \ . \tag{7}$$

Using this construction we can think of an object that transforms in the $(\frac{1}{2}, \frac{1}{2})$ representation as a 2×2 matrix that transforms as (7) from the right, and as ψ_R from the left. Write the matrix as

$$\begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} \tag{8}$$

and show that its components v^μ transform as a vector under the Lorentz group.

Hint: Write this matrix as

$$\begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} = \sum_{\mu=0}^{4} v_\mu \bar{\sigma}^\mu \ ,$$

where $\bar{\sigma} = (\mathbf{1}, -\sigma)$

Exercise 2 [*Majorana fermions*]: Let $\chi_a(x)$, $a = 1, 2$ be a two-component spinor that transforms in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, i.e. as ψ_L.

(i) Show that the equation

$$i\bar{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0 \ , \tag{9}$$

where $\bar{\sigma} = (\mathbf{1}, -\sigma)$ is relativistically invariant.

(ii) Show that

$$\sigma^\mu \bar{\sigma}^\nu \, \partial_\mu \partial_\nu = \Box \ ,$$

where $\sigma^\mu = (\mathbf{1}, \sigma)$. Hence deduce that equation (9) implies the Klein-Gordon equation, $(\Box + m^2)\chi = 0$.

Exercise 3 [*Series identity*]: Prove the identity

$$\sum_{n\in\mathbb{Z}} e^{ikn} = 2\pi \sum_{m\in\mathbb{Z}} \delta(k + 2\pi m) \ . \tag{10}$$