Effective Field Theories: Pions & Nucleons

Take $\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi_n)(\partial^\mu \varphi_n) - \frac{\mu^2}{2}\varphi_n\varphi_n - \frac{\lambda}{4}(\varphi_n\varphi_n)^2 + \dots$

$n = 1, 2, 3, 4$

with $\vec{\phi}$ isovector pseudoscalar field
and ϕ_4 isoscalar scalar field.

We would like to cast this Lagrangian in terms of Goldstone boson fields plus degrees of freedom "orthogonal" to Goldstone bosons

First, we observe that we can generate all fields $\phi_n(x)$ from one element in the field space

$$(0, 0, 0, \sigma)$$

using a symmetry rotation

$$\phi_n(x) = R_{n4}(x)\,\sigma(x)$$

We have

$$\sum_n \phi_n(x)^2 = \underbrace{\left(\sum_n R_{n4}(x)^2\right)}_{1} \sigma^2(x)$$

$$\rightsquigarrow \quad \sigma(x) = \sqrt{\sum_n \phi_n(x)^2}.$$

The field $\sigma(x)$ is invariant under the unbroken symmetries of the remaining group after SSB, and so is its vev. Thus it is not a Goldstone boson. Casting the Lagrangian in terms of $\sigma(x)$ we have:

$$\mathcal{L} = +\frac{1}{2}\sum_{n=1}^{4}\left(R_{n4}^{(x)}\,\partial_\mu\sigma(x) + \sigma(x)\,\partial_\mu R_{n4}(x)\right)^2$$

$$-\frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4$$

Using that $R(x)$ is a rotation,

$$\sum_n R_{n4}^2 = 1\,, \qquad \sum_n R_{n4}\,\partial_\mu R_{n4} =$$

$$= \frac{1}{2}\,\partial_\mu\left(\sum_n R_{n4}^2\right) = 0.\,,$$

we have:

$$\mathcal{L} = +\frac{1}{2}(\partial_\mu\sigma)(\partial^\mu\sigma) - \frac{1}{2}\sigma^2\sum_{n=1}^{4}(\partial^\mu R_{n4})(\partial_\mu R_{n4})$$

$$-\frac{\mu^2}{2}\sigma^2 - \frac{\lambda}{4}\sigma^4\,.$$

If $\mu^2<0 \sim \langle\sigma\rangle = \frac{|\mu|}{\sqrt{\lambda}}$. Instead of Φ_n we now have the degrees of freedom

$$\sigma' = \sigma - \frac{|\mu|}{\sqrt{\lambda}}$$ and the variables

which parameterize the rotation $R_{4\Omega}$

for example $\boxed{\mathcal{J}_\alpha = \frac{\phi_a}{\phi_a + \sigma}}$

$$R_{\alpha 4} = -R_{4\alpha} = \frac{2\mathcal{J}_\alpha}{1+\vec{\mathcal{J}}^2}$$

$$R_{44} = \frac{1-\vec{\mathcal{J}}^2}{1+\vec{\mathcal{J}}^2}$$

yielding

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - 2\sigma^2 \vec{D}_\mu \cdot \vec{D}^\mu$$
$$-\frac{1}{2}\mu^2\sigma^2 - \frac{\lambda}{4}\sigma^4$$

Where

$$\vec{D}_\mu \vec{\mathcal{J}} = \frac{\partial_\mu \vec{\mathcal{J}}}{1+\vec{\mathcal{J}}^2}$$

Notice that $\vec{\mathcal{J}}$ are massless degrees of freedom (no $\alpha \vec{\mathcal{J}}^2$ can be created.)
Expanding in $\frac{1}{1+\vec{\mathcal{J}}^2} = 1 - \vec{\mathcal{J}}^2 - (\vec{\mathcal{J}}^2)^2 \ldots$

we see that all interactions of $\vec{\mathcal{J}}$ particles vanish as the momenta go to zero.

The Lagrangian is invariant under $SO(4)$ transformations

For isospin rotations $\delta_{\text{UNBROKEN}} \vec{J} = \theta \times \vec{J}$

$$\delta \sigma = 0$$

this is obvious. We must work a bit ~~We know~~ further for the transformations of the broken generators. In terms of original fields ϕ_4,

$$\delta \vec{\phi} = 2 \vec{\epsilon} \phi_4 \quad , \quad \delta \phi_4 = -2 \vec{\epsilon} \cdot \vec{\phi} \quad . \text{ But}$$

$$J_\alpha = \frac{\phi_\alpha}{\phi_4 + \sigma}$$

$$\sim \quad \delta_{\text{BROKEN}} \vec{J} = \epsilon (1 - \vec{J}^2) + 2 \vec{J} (\vec{\epsilon} \cdot \vec{J}) , \; \delta \sigma = 0 .$$

This is a <u>non-linear</u> transformation for the field $\vec{J}$. However, we can deduce that

$$\delta \vec{D}_\mu = 2 (\vec{J} \times \vec{\epsilon}) \times \vec{D}_\mu$$

transforms <u>linearly</u> (although with a field dependent transformation).

General Broken Global Symmetries

Let's assume a SSB pattern

$$G \longrightarrow H \quad , \quad H \subset G$$

Lagrangian symmetry group — vacuum expectation value's symmetry group.

$\mathcal{L}$ is invariant under

$$\psi_n(x) \longrightarrow \sum_m g_{nm} \psi_m(x)$$

where the group parameters g_{nm} are global $\frac{\partial g_{nm}}{\partial x^\mu} = 0$.

Vacuum invariance under H, means that

$$\sum_m h_{nm} \langle \psi_m \rangle = \langle \psi_n \rangle \quad \forall h \in H.$$

Broken global symmetries result to Goldstone bosons. Let's assume that we are working with a real representation of G. The mass matrix of the theory has zero eigenvalues for the eigenvectors

$$\sum_m [T^\alpha]_{nm} \langle \psi_m \rangle \;\; (= \delta\langle \psi_n \rangle)$$

[If T^α is a broken generator $\delta\langle \psi_n \rangle \neq 0$]

We can define/find the fields which are orthogonal to Goldstone bosons by requiring that

$$\underbrace{\left(\tilde{\psi}(x) \right)}_{\substack{\text{fields} \\ \text{free of Goldstones}}} \cdot \underbrace{\left(T^a \langle \psi \rangle \right)}_{\substack{\text{zero mass} \\ \text{eigenstate of} \\ \text{vev's.}}} = 0$$

or, precisely,

$$\sum_{nm} \tilde{\psi}_n(x) \, T^{\alpha}_{nm} \langle \psi_m \rangle = 0 .$$

We shall prove that starting from a generic configuration $\psi_n(x)$, we can obtain fields $\tilde{\psi}_n$ by applying a "local" group transformation.

$$\psi_n(x) = \sum_m \gamma_{nm} \tilde{\psi}_m(x)$$

with

$$\gamma \in G .$$

Let's consider the quantity

$$V_\psi(g) = \sum_{nm} \psi_n \, g_{nm} \langle \psi_m \rangle ,$$

where g_{nm} runs over the whole group G.

This is a continuous function of g. For compact groups G, $V_\psi(g)$ is also bounded. Let's assume that for $g = \gamma(x)$

$V_{\psi(x)}(g)$ is at its maximum for every point x in space-time. Thus

$$\delta V_{\psi(x)}(\gamma(x)) = 0$$

$$\longrightarrow \sum_{nm} \psi_n (\delta\gamma_{nm}) \langle\psi_m\rangle = 0$$

But, $\delta\gamma = i\sum_\alpha \epsilon^\alpha T^\alpha \gamma$

or

$$\delta\gamma_{nm} = i\sum_\alpha \epsilon^\alpha \gamma_{nl} T^\alpha_{lm}$$

which yields:

$$0 = i\sum_\alpha \epsilon_\alpha [\gamma^{-1}(x)]_{ln} \psi_n(x)\, t^\alpha_{lm} \langle\psi_m\rangle$$

So Eq. 1 is satisfied for

$$\tilde\psi(x) = \gamma^{-1}(x)\,\psi(x) \qquad \text{where } \gamma(x)$$

is such that $V_\psi(g)$ is maximum $\forall\, x$, for $g = \gamma$.

What it remains is to rewrite $\varphi(x) = \gamma(x)\,\tilde{\varphi}(x)$ for the fields in the original Lagrangian. If $\gamma(x)$ was constant in space-time (global) this rewriting would let the Lagrangian invariant. But now $\gamma(x)$ differs from point to point. We can write,

$$\gamma(x) = \gamma(x_0) + \partial_a \gamma(x_0)\,(x^a - x_0^a) + \cdots$$

$$\rightsquigarrow \gamma(x) = \gamma(x_0) + F[\partial_\mu \gamma(x)],$$

wher F can be ~~an~~ differential operator.

The Lagrangian

$$\mathcal{L}[\gamma(x)\,\tilde{\varphi}(x)] = \mathcal{L}(\overset{\text{global}}{\gamma(x_0)}\,\tilde{\varphi}(x)]$$

$$+ \text{ function of } \{\text{derivations of } \gamma(x),\ \tilde{\varphi}(x)\}$$

$$= \mathcal{L}(\tilde{\varphi}(x)) + \text{function of } \partial_\mu \gamma(x) \text{ and higher derivations}$$

$\rightsquigarrow$ Rewriting φ as $\tilde{\varphi}$ orthogonal to Goldstone modes and Goldstone modes $\gamma(x)$, results to a Lagrangian which is a functional of $[\partial_\mu \gamma(x)]$?

This forbids terms lacking derivatives

so

$$\sim m_B^2 \, B(x) \, B(x).$$

We can also predict that at low energies Goldstone interactions vanish:

since $\partial_\mu B \rightarrow p_B^\mu$ in the Feynman rules.

Spontaneously Broken Local Symmetries

The discussion of before changes dramatically if the symmetry group ~~G~~ G is local. Then, after SSB, we can find

$$\tilde{\psi}(x) = \gamma^{-1}(x)\,\psi(x)$$

which satisfy

$$\tilde{\psi}\cdot\left(T^a\cdot\langle\psi\rangle\right) = 0$$

but the rewriting of $\mathcal{L}$ in terms of $\tilde{\psi}$ (rather than ψ) let's $\mathcal{L}$ invariant. The Goldstone fields

in $\gamma(x)$ disappear completely from

the spectrum ! We have found an exception of paramount importance for the statement of the Goldstone theorem. In case of local symmetry transformations Goldstone fields are not physical degrees of freedom, and can be made to dissappear with a clever gauge-fixing:

$$\tilde{\psi}\cdot\left(T^a\langle\psi\rangle\right) = 0$$

Any Lagrangian which is locally invariant under a continuous symmetry transformation requires gauge bosons, for forming covariant derivatives. Let's look at quadratic terms

$$\mathcal{L} \ni \frac{1}{2}\sum_{n}\left(\partial_\mu \phi_n + ig\sum_{m,\alpha} T^\alpha_{nm} A^\alpha_\mu \phi_m\right)^2$$

and shift $\phi_n = \langle\phi_n\rangle + \phi_n' = v_n + \phi_n'$

Then

$$\mathcal{L} \ni \frac{1}{2}\sum_n (\partial_\mu \phi_n')^2 + \frac{1}{2}\sum_{\alpha\beta} \mu^2_{\alpha\beta} A_{\alpha\mu} A^\mu_\beta$$

with $\mu^2_{\alpha\beta} \equiv g^2 \sum_{nm} T^\alpha_{nm} T^\beta_{n\ell} v_\ell v_m$

The mass matrix of gauge bosons is proportional to couplings!

T^α_{nm} is imaginary for real group reps (we have assumed ϕ to be real).

$\rightarrow \quad \mu^2_{\alpha\beta} \geq 0$.

~~Let's assume~~ If there is a real linear combination of generators which does not break the symmetry:

$$\left(\sum_\alpha c_\alpha T^\alpha\right) \cdot v = 0$$

$$, \sim \quad \boxed{\sum_\alpha c_\alpha T^\alpha_{nm} v_m = 0}$$

then,

$$\sum_\beta c_\beta \mu^2_{\alpha\beta} =$$

$$= -g^2 \sum_\beta c_\beta \sum_{n m l} c_\alpha T^\alpha_{n\beta} t^\beta_{nl} v_m v_l =$$

$$= -g^2 \sum_n \left(\sum_{\alpha, m} c_\alpha T^\alpha_{nm} v_m \right)^2 \dots = 0.$$

If we have a massless gauge boson,

the $c^T M^2 c \geq 0$.

$$\sim \sum_{\alpha\beta} c_\alpha c_\beta \mu^2_{\alpha\beta} = \sum_\alpha \left(\sum_{\alpha, m} c_\alpha \cdot i T^\alpha_{nm} v_m \right)^2 \geq$$

requires that "=" is satisfied if $\underbrace{\left(\sum_\alpha c_\alpha T^\alpha_{nm} \right) \cdot v_m}_{\Downarrow} =$

unbroken GENERATOR.

Let's now look at ~~the~~ the propagator of gauge bosons.

$$\frac{1}{2} \sum_\alpha (\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu})^2 - \frac{1}{2} \sum_{\alpha\beta} \mu^2_{\alpha\beta} A_{\alpha\mu} A_{\beta}^{\mu}$$

$$= \frac{1}{2} \sum_{\alpha\beta} A_\alpha^\nu \, \Delta_{\alpha\nu, \beta\mu}(\partial) \, A_\beta^\mu + \text{total derivative}.$$

with $\Delta_{\alpha\mu, \beta\nu} \equiv \delta_{\alpha\beta} \left[g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu \right] - \mu^2 g_{\alpha\nu}$

In momentum space,

$$\Delta^{\mu\nu,\alpha\beta}(k) = \frac{\delta_{\alpha\beta}}{(k^2 - \mu^2)} \left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2} \right]$$

~~The $k^\mu k^\nu$~~ At $k \to \infty$.

$$\Delta^{\mu\nu,\alpha\beta} \to \text{Constant}.$$

This spoils renormalizability arguments, which are based on power count.
In massless theory, after gauge fixing,

$$\Delta(k) \sim \frac{1}{k^2} \left[-g_{\mu\nu} + \frac{k^\mu k^\nu}{k^2} \right].$$