Example of a spontaneously broken global symmetry

Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_n (\partial_\mu \varphi_n)^2 - \frac{M^2}{2}\sum_n \phi_n^2 - \frac{g}{4}\sum_n \phi_n^4$$

of N real scalar fields ϕ_n, $n = 1 \ldots N$.
The Lagrangian is symmetric under rotations

$$\phi_n \to \phi_n' = R_{nm}\,\phi_m \quad , \quad R^2 = 1$$

and the symmetry group is $O(N)$

The field vacuum expectation values can be computed by minimizing the effective potential

$$\left.\frac{\partial V_{eff}}{\partial \phi_\ell}\right|_{\phi_\ell = \langle\phi_\ell\rangle} = 0 .$$

Ignoring loop effects, the effective potential is equal to the classical potential.

$$V_{eff}(\phi) \simeq \frac{M^2}{2}\sum_n \phi_n^2 + \frac{g}{4}\sum_n \phi_n^4 .$$

and its extrema are solutions of

$$\langle\phi_\ell\rangle\left(M^2 + g\sum_n \langle\phi_n\rangle^2\right) = 0.$$

For $\mu^2, g > 0$, we find a minimum

$$\langle \phi_\ell \rangle = 0$$

which is invariant under $O(N)$ rotations.

Spontaneous symmetry breaking occurs for $\mu^2 < 0$ & $g > 0$. The minima are found to satisfy:

$$\boxed{\sum_n \langle \phi_n \rangle^2 = -\frac{\mu^2}{g}} \quad (\text{Eq. }1)$$

and they are degenerate. Only one is chosen (spontaneously), but it could be any of the solutions of the above equation.

Let us look at the spectrum of the theory and specifically the masses of the scalar particles. These can be given by diagonalizing the quadratic terms in the effective potential. The corresponding "mass-matrix" is

$$D^{-1}_{nm}(0) \simeq \left.\frac{\partial^2 V}{\partial \phi_n \partial \phi_m}\right|_{\phi = \langle \phi \rangle} + (\text{loop-corrections})$$

$$= \underbrace{\mu^2 \delta_{nm} + g\, \delta_{nm} \sum_\ell \langle \phi_\ell \rangle^2}_{\emptyset \ (\text{Eq. }1)} + 2g \langle \phi_n \rangle \langle \phi_m \rangle$$

as
$$M_{nm}^2 = 2g \langle\phi_n\rangle\langle\phi_m\rangle$$

or in a matrix-form:

$$M_{nm}^2 = 2g \begin{pmatrix} \langle\phi_1\rangle^2 & \cdots & \cdots & \langle\phi_n\rangle\langle\phi_1\rangle \\ \langle\phi_1\rangle\langle\phi_2\rangle & \langle\phi_2\rangle^2 & & \\ \vdots & & \ddots & \\ \langle\phi_1\rangle\langle\phi_n\rangle & \cdots & \cdots & \langle\phi_n\rangle^2 \end{pmatrix}$$

To diagonalize and find the mass eigenvalues, we have:

$$\det\left(M_{nm}^2 - \mu^2\delta_{nm}\right) = 0 \rightsquigarrow$$

$$\rightsquigarrow \quad (\mu^2)^{N-1}\left(\mu^2 - 2g\sum_i \langle\phi_i\rangle^2\right) = 0 .$$

We find one massive particle with a mass

$$\mu^2 = 2g\sum_i \langle\phi_i\rangle^2 = 2|M|^2 . \qquad \text{from (Eq. 1)}$$

and

(N-1) Goldstone boson particles with zero mass.

The original group $O(N)$ has

$\frac{1}{2}N(N-1)$ generators.

The spontaneous breaking picks us one

direction in the space of $\langle\phi_i\rangle$, $i=1,\ldots,N$. The linear combinations of fields which are orthogonal to this direction can be freely rotated to each other. The vev's have a remaining $O(N-1)$. The number of generators which have been <u>broken</u>* is:

$$\frac{1}{2}N(N-1) - \frac{1}{2}(N-1)(N-2) = N-1 .$$

We will get back to this soon.

Let's now go back to the second proof of Goldstone theorem, where we have found that the Goldstone boson states yield

$$\int d^3\vec{p}_B \langle 0 | J^\mu(x) | B\rangle\langle B | \phi_n(y) | 0\rangle \, \delta(p-p_B) =$$

$$= \frac{i}{(2\pi)^3} \rho_n(p^2)\cdot p^\mu$$

From Lorentz invariance we infer that

$$\langle 0 | J^\mu(x) | B\rangle = \frac{i\, p_B^\mu F \, e^{i p_B\cdot x}}{(2\pi)^{3/2}\sqrt{2 p_B^0}}$$

and

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* A broken symmetry generator transforms the vacuum state to a different one.
An unbroken generator leaves the vacuum invariant.

$$\langle B | \phi_\eta(x) | \underline{0} \rangle = \frac{Z_\eta \, e^{-i p_B \cdot x}}{(2\pi)^{3/2} \sqrt{2 p_B^0}}$$

which yields

$$\rho_\eta(m^2) = F \cdot Z_\eta \, \delta(m^2)$$

We had also shown that

$$\langle [J^0(\vec{y}, t), \phi_\eta(\vec{x}, t)] \rangle = i \delta^{(3)}(\vec{x} - \vec{y}) \int \rho_\eta(m^2) \, dm^2$$

and integrating over $\vec{y}$,

$$\langle \underbrace{[Q, \phi_\eta(\vec{x}, t)]}_{\delta\phi_\eta} \rangle = i F \cdot Z_\eta$$

$$\leadsto \quad \langle \delta \phi_\eta \rangle = i \cdot F \cdot Z_\eta .$$

In the prese of multiple broken generators T^a, we have: an equal number of currents $J^\mu_{(a)}$ and therefore Goldstone bosons.

This is because we can repeat the same proof for each one of the $J^{(\cdot)}_\mu$ currents.

and we have:

$$\langle B_a | \phi_n(x) | 0 \rangle = \frac{Z_{an}\, e^{-ip_B\cdot x}}{(2\pi)^{3/2}\sqrt{2p_B^0}}$$

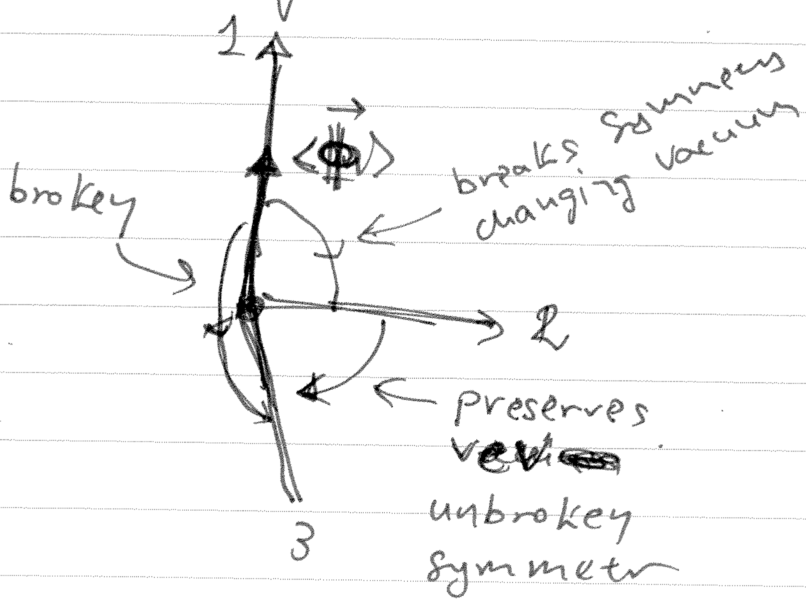
$$\langle 0 | J_a^{\mu(0)} | B_\beta \rangle = \frac{i\, F_{a\beta}\, e^{+ip_B\cdot x}}{(2\pi)^{3/2}\sqrt{2p_B^0}} \cdot p_B^\mu .$$

This yields the sum-rule:

$$i \sum_b F_{ab} Z_{bn} = -\sum_m T^a_{nm} \langle \phi_m \rangle ,$$

where $i\epsilon\, \delta\phi^{(\alpha)} = -i\epsilon \sum_m T^\alpha_{nm} \phi_m$.

In our example of $O(N)$ symmetric Lagrangian, we can choose a vacuum aligned to one of the generators



$\langle \phi_n \rangle = v\, \delta_{a1}$

Let's take "α" to correspond to a broken generator.

$\alpha \in 2..N$ corresponding to rotations which leave the "a"-axis invariant.

They we have:

$$i \sum_b F_{ab} Z_{bn} = -v (T_a)_{n1}$$

Exercise:

We can write the generators $(T_a)_{n1}$

as $T^{\alpha}_{a1} = -T^{\alpha}_{1a} = -i$ for $n = a$

and $T^{\alpha}_{n1} = -T^{\alpha}_{1n} = 0$ for $n \neq a$.

Prove it for $O(3) \to O(2)$.

They we have:

$$\sum_b F_{ab} Z_{b\alpha} = v \quad , \quad a = 2 \ldots N.$$

For $b \neq a$, we have

$F_{ab} = \delta_{ab} F$ (To prove)

& $Z_{\alpha b} = \delta_{\alpha b} Z$. and we can choose

$Z = 1$, by renormalizing the field ϕ.

We then have

$$F = v.$$

So $F \sim \langle 0 | J^{a}_{\alpha} | B_b \rangle$ is associated with

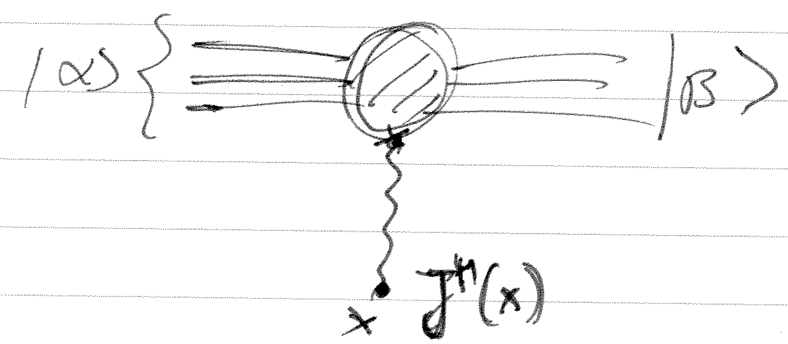
with the strength of electroweak Symmetry breaking.

Interactions of Goldstone Bosons

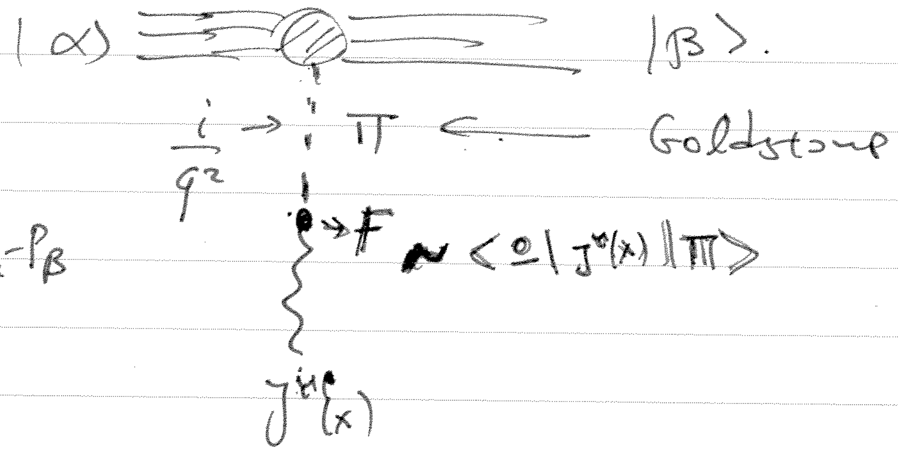
What is the probality to produce a Goldstone boson in a certain interaction?

To compute this, we consider a matrix-element

$$\langle \beta | J^\mu(x) | \alpha \rangle$$



A Goldstone boson has the same quantum numbers as J^μ and it therefore couples to **it**. If we were to write a Feynman diagram it would be something like



$q \equiv p_\alpha - p_\beta$

we can trivially predict the dependence of this amplitude in x;

$$\langle \beta | J^\mu(x) | \alpha \rangle = e^{-i(p_\alpha - p_\beta)} \langle \beta | J^\mu(0) | \alpha \rangle .$$

The matrix element $\langle \beta | J^\mu(0) | \alpha \rangle$ has unknown Feynman rules and more Feynman diagrams than what is drawn above. But given that the Goldstone boson is a scalar, we can infer:

$$\langle \beta | J^\mu(0) | \alpha \rangle = \frac{i}{q^2} F M_{\beta\alpha} + N_{\beta\alpha}$$

where $M_{\beta\alpha}$ is the transition amplitude from $|\alpha\rangle \to |\beta\rangle$ (~~without conserving momentum~~) ~~when $q \neq 0$~~ and emitting a Goldstone. The second term does not involve resonant diagrams as $q^2 \to 0$ with a Goldstone boson propagator. (It may have other $\frac{1}{q^2}$ singularities as we shall explain).

Multiplying with q^μ and taking $q^\mu \to 0$, we have

$$\langle \beta | \overbrace{q\cdot J(0)}^{\partial_\mu J^\mu = 0} | \alpha \rangle = q\cdot N_{\beta a} + i\, F \cdot M_{\beta\alpha}$$

$$\Rightarrow \quad \boxed{M_{\beta\alpha} = \frac{i}{F}\, q\cdot N_{\beta\alpha}}$$

$M_{\beta\alpha}$ is an amplitude to produce a Goldstone boson.

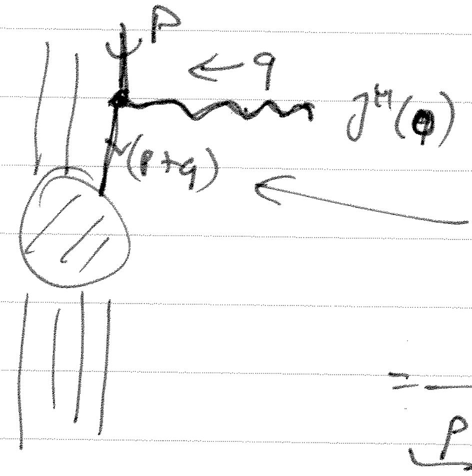
$N_{\beta a}$ is an amplitude without

Goldstone boson vertices to the current $J^\mu(0)$.

Let's take the limit of a soft Goldstone boson:

$$q \to \phi.$$

Then $M_{\beta a}\Big|_{q\to 0}$ is made out of the poles residue of the poles of $N_{\beta a}$



$$\frac{i}{(p+q)^2 - m^2} = \frac{i}{\underbrace{p^2 - m^2}_{p^2 = m^2} + 2p\cdot q} = \frac{i}{2p\cdot q}.$$

Notice that only insertions in external legs can produce soft singularities. (!)

So, the amplitude $M_{\beta a}$ to emit a Goldstone boson in the transition

$$|\alpha\rangle \rightarrow |\beta\rangle.$$

is given by the same amplitude

$$|\alpha\rangle \rightarrow |\beta\rangle$$ without emitting a Goldstone boson, by modifying the Feynman rules for external legs, inserting the current $J^{q}(0)$.

(!) The coupling constant associated with the emission of a Goldstone boson is

$\frac{1}{F}$ the inverse of the

strength of spontaneous symmetry breaking.