Notes on "Effective non-renormalizable theories at one-loop" by Mary K. Gaillard

1. Effective non-renormalizable theories in physics

1.1 Infrared limits of the standard model

- Fermi theory of β-decay describes weak charged interactions in the limit of small momentum transfer

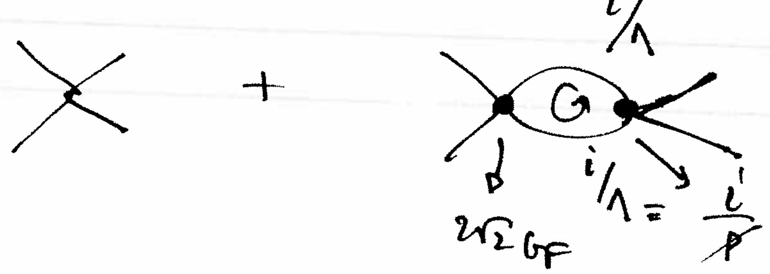
$$|q^2| \ll m_W^2 = \frac{2\sqrt{g}}{8\, G_F}$$

- $SU(2)_L \otimes SU(2)_R$ chiral invariant σ-model describes pion dynamics at low energies. In this case, we cannot find a parameter of QCD which yields this model at a certain limit.

Take the Fermi theory at tree-level:

$$\mathcal{L}_{tree} = 2\sqrt{2}\, G_F \, (\bar{\psi}_L \gamma^a \psi_L)(\bar{\psi}_L \gamma_a \psi_L)$$

and start computing loop corrections



Loop $= \int \frac{d^4 q}{(2\pi)^4} \cong \Lambda^4$

This gives

$$\mathcal{L}_{eff}^{1-loop} \sim \frac{1}{16\pi^2}\Lambda^2 (2\sqrt{2}\, G_F)^2 (\bar\psi_L \gamma \psi_L)^2$$

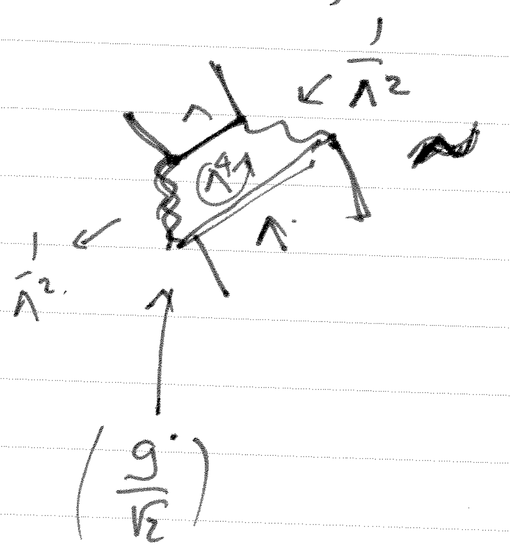
$$\sim \frac{G_F^2}{2\pi^2}\Lambda^2 (\bar\psi_L \gamma \psi_L)^2$$

In ~~the~~ ~~SM~~: identify, $\Lambda^2 \approx M_W^2 = \sqrt{2}\, g^2 / 8 G_F$

$$\mathcal{L}_{eff} = g^2 \frac{\sqrt{2}\, G_F}{16\pi^2} (\bar\psi_L \gamma \psi_L)^2 = \frac{\alpha}{8\pi} \mathcal{L}_{tree}.$$

$\alpha \equiv \frac{g^2}{4\pi}$ weak "fine structure" constant.

Let's do the "exact" thing in the SM theory

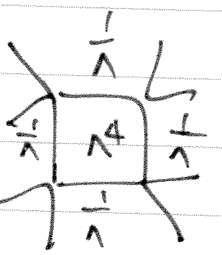


$$\sim \frac{1}{16\pi^2}\left(\frac{g}{\sqrt{2}}\right)^4 \frac{1}{M_W^2} (\bar\psi_L \gamma \psi_L)^2$$

$$\sim \frac{\alpha}{8\pi} \mathcal{L}_{tree}$$

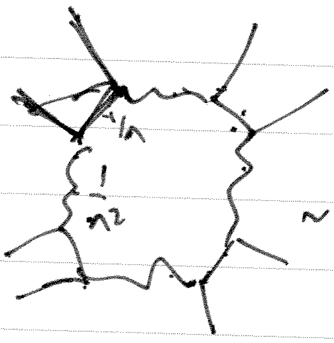
Notice that given that the one loop

correction is proportional to tree, f. it can be absorbed into a redefinition of the Fermi constant. But, let's look at more complicated processes:



$$\sim \frac{1}{16\pi^2}\left(2\sqrt{2}G_F\right)^4 \ln\left(\frac{\Lambda^2}{\mu^2}\right)\left(\bar{\psi}_L\gamma_\mu\psi_L\right)$$

M is a fermion mass or an external momentum (q_{ext}). Let's do the same in the SM. This is a new approach



$$\sim \frac{\Lambda^4}{(\Lambda^2)^4\,\Lambda^4} \sim \frac{1}{m_W^8}$$

$$\mathcal{L}_{eff} \sim \frac{1}{16\pi^2}\left(\frac{g}{\sqrt{2}}\right)^8 \frac{1}{M_W^8} \ln\left(\frac{m_W^2}{\mu^2}\right)\left(\bar{\psi}_L\gamma\psi_L\right)$$

Comparing the two, we deduce that

$$\Lambda \sim O(m_W)$$

New physics, W-boson, enters to damp the divergences of the effective theory.

1.2 The large Higgs mass limit of the Standard model

Neglecting gauge interactions

$$\mathcal{L}_{SM} \xrightarrow[g\to 0]{} \mathcal{L}_H = \partial_\mu \varphi \partial^\mu \varphi^* - \lambda\left(|\varphi|^2 - \frac{v^2}{2}\right)^2$$

(Eq.-1)

This is invariant under $SO(4)$ (or $SU(2)\times SU(2)$) linear transformations among the components of

$$\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\pi_1 + \pi_2 \\ \sigma - i\pi_3 \end{pmatrix}$$

Explicitly,

$$\mathcal{L}_H = \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2}\partial_\mu \vec{\pi}\cdot\partial^\mu \vec{\pi} -$$

$$-\frac{1}{4}\lambda\left(\sigma^2 + \vec{\pi}^2 - v^2\right)$$

~~Not all~~ To separate the Goldstone degrees of freedom explicitly, we can write:

$$\varphi = \frac{1}{\sqrt{2}} e^{i\frac{\vec{\theta}\cdot\vec{\tau}}{v}} \begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

$$\rightsquigarrow \mathcal{L}_H = \mathcal{L}_{KE}(\theta,\rho) - \frac{1}{4}\left(\rho^2 - v^2\right)^2$$

The two formulations give of course identical results for S-matrix elements, it is

$$v = \langle \rho \rangle = \langle \sigma \rangle = \sqrt{2}\,\langle |\phi| \rangle$$

$$\rho = \sigma + O\left(|\varphi| - \frac{v}{\sqrt{2}}\right)^2$$

$$\theta_i = \pi_i + O\left(|\varphi| - v/\sqrt{2}\right)^2$$

Let's switch on now gauge interactions.

$$\partial_\mu \varphi \longrightarrow D_\mu \varphi = (\partial_\mu + i A_\mu)\varphi$$

$$A_\mu = \frac{g}{2} T_\alpha A_\mu^\alpha$$

T^α are generators of the $SU(2)_L \times U(1)$ group and A_μ^α gauge fields.

The gauged Lagrangian is invariant under

$$\varphi' = U(x)\,\varphi,$$

$$A_\mu' = U A_\mu U^{-1} + i(\partial_\mu U)\,U^{-1}$$

$$\mathcal{L}(\varphi', A_\mu') = \mathcal{L}(\varphi, A_\mu)$$

We can choose a <u>unitary gauge</u>:

$$u = e^{-\frac{i\vec{\theta}\cdot\vec{\tau}}{v}}$$

$$\varphi' = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \rho \end{pmatrix}$$

The Lagrangian in this gauge is

$$\mathcal{L}_u = \mathcal{L}(A, \rho)$$

and the physical Higgs boson is

$$H = \rho - v .$$

This gauge is good for demonstrating the physical fields, but not so good for loop calculations and dimensional analysis.

$$\mathcal{L}_R = \mathcal{L}(A, \sigma, \pi) \quad , \quad H \equiv \sigma - v.$$

In either case we find ~~that~~

$$m_H^2 = 2v^2\lambda.$$

The vacuum expectation value can be determined from experiment:

$$D_\mu \varphi \, D^\mu \varphi = \frac{g^2 v^2}{4} \left(W_\mu^+ W^{\mu -} + \frac{1}{2\cos^2\theta_W} Z_\mu Z^\mu \right)$$

$$+ \, m_W W_\mu^+ \partial^\mu \pi^- + \cdots$$

$$v^2 = \frac{4 m_W^2}{g^2} = (\sqrt{2} G_F)^{-1} \approx \left(\frac{1}{4}\,\mathrm{TeV}\right)^2$$

~~Low energy e~~ Therefore, the $m_H \to \infty$ limit corresponds to $\lambda \to \infty$.

For the potential energy term

$$\lambda (\sigma^2 + \vec{\pi}^2 - v^2)^2$$

to remain finite, the only possibility is

$$\sigma = \sqrt{\vec{\pi}^2 + v^2}$$

or equivalently

$$\rho^2 = v^2.$$

The variable ρ (or σ) can be eliminated from the Lagrangian. The constraint

$$\varphi^2 = (\vec{\pi}^2 + \sigma^2) = v^2$$

is invariant under ~~linear local~~ $SO(4)$ or $SU(2) \times SU(2)$ ~~transformations.~~ . Upon imposing it, the linear transformation.

$$\delta \pi_i = G_{jk}\, a_j\, \pi_k + \beta_i\, \sigma,$$

$$\delta \sigma = - \beta_i\, \pi_i$$

become $\delta \pi_i = G_{ijk}\, \alpha_j\, \pi_k + \beta_i\, (\pi^2 - v^2)^{1/2}$

which is non-linear. The Lagrangian ~~will~~ becomes

$$\mathcal{L} \ni \frac{1}{2}\, \partial_\mu \pi^i\, \partial^\mu \pi^j \quad g_{ij}$$

$$g_{ij} = \delta_{ij} + \frac{\pi_i \pi_j}{v^2 - \vec{\pi}^2} \qquad \text{(scalar metric)}$$

1.4 The gauged non-linear σ-model.

Let's define

$$R^i_{jkl} = \frac{1}{v^2}\left(\delta^i_l\, g_{jk} - \delta^i_k\, g_{jl}\right)$$

and the Ricci tensor

$$R_{ij} = \frac{(1-N)}{v^2}\, g_{ij} \quad \text{, with}$$

$N \equiv$ # of real scalars π_i. The one-loop effective action can be computed and we obtain:

$$\mathcal{L}_{eff} = \frac{1}{2} g_{ij} \partial^{\mu} \pi^i \partial_{\mu} \pi^j \left(1 - \frac{(N-1)}{16\pi^2} \frac{\Lambda^2}{v^2}\right)$$

$$+ \frac{1}{64\pi^2} \mathrm{Tr}\left(R^2 + \frac{1}{3} G_{\mu\nu} G^{\mu\nu}\right) \ln\left(\frac{\Lambda^2}{?}\right) + \ldots$$

The first term renormalizes the

pion fields and the vev.

$$\pi_R = Z\pi$$

$$v_R = Z v$$

$$Z^2 = 1 - \frac{(N-1)}{16\pi^2} \frac{\Lambda^2}{v^2}$$

The logarithm is dimensionless. But in the massless σ-model, there is no scale parameter to scale the cutoff. Similar situation as in QCD where a Λ_{QCD} emerges after a resummation of infrared divergent contributions.

Then

$$\mathcal{L}_{eff} \to \frac{1}{2} g_{ij} \partial^{\mu}\pi_i \partial_{\mu}\pi_j Z^2$$

$$+ \frac{1}{64\pi^2} \mathrm{Tr}\left[R\left(\ln\frac{\Lambda^2}{\partial^2} + a\right)R + \frac{1}{3} G_{\mu\nu}\left(\ln\frac{\Lambda^2}{\partial^2} + a'\right)G^{\mu\nu}\right] +$$

$+ O(\partial/\Lambda)$. a & α' are unknown constants of $O(1)$.

We can go on and compute the amplitude of scattering amplitudes

$$\mathcal{M}(\pi^+\pi^- \to \pi^+\pi^-) = -\frac{i4}{v^2} +$$

$$+ \frac{1}{64\pi^2 v^2}\left[3s^2 \ln\left(\frac{\Lambda^2}{-s}\right) + 3t^2 \ln\left(\frac{\Lambda^2}{-t}\right)\right.$$

$$\left. + 2u^2 \ln\left(\frac{\Lambda^2}{-u}\right) + \underbrace{- - -}_{\text{similar terms}} \right].$$

Recall that the matrix-elements for Goldstone boson scattering ~~and currents correspond to~~ can be obtained by replacing them with the currents that they are coupled to. Based on this we can derive "equivalence theorem".

S-matrix elements for (W_L, Z_L) scattering can be obtained up to correction of $O\left(\frac{m_W}{E_W}\right)$ $O\left(\frac{m_Z}{E_Z}\right)$ by replacing $W_L^\pm, Z_L$ with $\pi^\pm, \pi^0$.

Intuitively, a transformation

$$W_a \to W'_a + \partial_a \eta \quad \text{introduces}$$

a longitudinal component ($\partial_a \eta$) into the gauge field. So, we anticipate that

e.g. $W_L W_L \to W_L W_L$
grows with energy.

How can the divergences of the theory dumped?

Let's do the same computation with SM in $m_H \to \infty$ (forget gauge interaction for a moment). We obtain the same result as before, but with Λ being now the renormalization scale.

The SM effective action at one-loop is at $m_H \to \infty$

$$\mathcal{L}_{eff}^{1\text{-loop}} = -\frac{\Lambda^2}{16\pi^2}|D_a\varphi|^2 + \frac{\ln \Lambda^2}{16\pi^2}$$

$$\left(a F_{\mu\nu}F^{\mu\nu} + b|D_a\varphi|^2 + c(\varphi^\dagger D_a\varphi)^2 + \dots\right) + \text{finite terms}$$

[illegible]

In the unitary gauge

$$(\varphi^\dagger D_\alpha \varphi)^2 \Big|_{\pi=0} = \frac{-g^2 v^2}{16 \cos\theta_W} Z_\mu Z^\mu)$$

There is a divergent term, breaking SO(4) invariance, but preserving SU(2) × U(1)(-

which contributes a shift to the Z-boson mass.

$$\rho - 1 \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} - 1 =$$

$$= \frac{-3g^2}{64\pi^2} \tan^2\theta_W \ln\left(\frac{\Lambda^2}{m_W^2}\right) + \text{finite}.$$

ρ is well measured and very close to 1. This gives a $\Lambda < 3$ TeV

If we put $\Lambda^2 = m_H^2$, we get the leading log term in the SM calculation.

Custodial symmetry cannot be broken seriously. (experimental result).

The SM with a ~~Hig~~ Higgs boson is OK [illegible] $|\rho - 1|$ small. But other models must — satisfy this.