The Physics of Electroweak symmetry breaking Exercise sheet 1

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Exercise 1.1 Review of the quantum effective action

In the lecture, you will see a proof to Goldstone's theorem that makes use of the *quantum effective action*. We will have a little review on its definition and properties.
We start from the path integral of a scalar field theory (also called the *generating functional*),

$$Z[J] = \int \mathcal{D}\phi \, e^{i\left(S[\phi]+\int \mathrm{d}^4x J(x)\phi(x)\right)} \,, \tag{1}$$

where $S[\phi] = \int \mathrm{d}^4x \, \mathcal{L}\,(\phi(x))$ stands for the classical action and $J(x)$ is an auxiliary external current. Z is the sum of all possible Feynman diagrams governed by the theory in the presence of the external current J. For $J = 0$, it is the all-order vacuum-vacuum amplitude.
N-point correlation functions can be obtained from Z by N-fold functional differentiation w.r.t. the current J:

$$\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle = \frac{1}{Z[J]}\frac{1}{i^n}\frac{\delta^n Z[J]}{\delta J(x_1)\ldots J(x_n)}\bigg|_{J=0} \,. \tag{2}$$

More convenient to work with is the functional $iW[J]$, which is the sum of all possible *connected* Feynman diagrams. Differentiating it generates only *connected* N-point-functions:

$$Z[J] = e^{iW[J]}\,; \quad \langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle_{\text{connected}} = \frac{1}{i^{n-1}}\frac{\delta^n W[J]}{\delta J(x_1)\ldots J(x_n)}\bigg|_{J=0} \,. \tag{3}$$

From this, we define the quantum effective action as

$$\Gamma[\langle\phi\rangle_J] := W[J] - \int \mathrm{d}^4x \, J(x)\langle\phi\rangle_J(x) \quad \text{with} \quad \langle\phi\rangle_J(x) := \frac{\delta W[J]}{\delta J(x)} \tag{4}$$

a) Derive the fact that the full generating functional iW can be obtained by replacing the classical action S with the quantum action Γ and keeping only *tree diagrams*:

$$iW[J] = \int\limits_{\substack{\text{connected}\\ \text{trees}}} \mathcal{D}\langle\phi\rangle_J \, e^{i\left(\Gamma[\langle\phi\rangle_J]+\int \mathrm{d}^4x\, J(x)\langle\phi\rangle_J(x)\right)} \tag{5}$$

Hints: Follow Weinberg, chapter 16.1, pp. 66:

- Introduce a factor $1/g$ in the exponent of the path integral e^{iW_Γ}, where W_Γ is $W[J]$ with S replaced by Γ.
- Use this parameter g to organise W_Γ as a loop-expansion.
- Take the limit $g \to 0$, i.e. keep only the tree diagrams.
- "Rediscover" the full $W[J]$ in this by using the relation $\frac{\delta\Gamma[\langle\phi\rangle_J]}{\delta\langle\phi\rangle_J(x)} = -J(x)$.

b) Derive the fact that the twofold derivative of the effective action is equal to the *inverse of the full propagator*:

$$\frac{1}{i}\frac{\delta^2\Gamma}{\delta\langle\phi\rangle_J(x)\delta\langle\phi\rangle_J(y)} = \left(\frac{1}{i}\frac{\delta^2 W}{\delta J(x)\delta J(y)}\right)^{-1} =: \Delta^{-1}(x-y) \tag{6}$$

Hints: Differentiate eq. (4) first w.r.t. the field VEV $\langle\phi\rangle_J(x)$ (as seen in the lecture) and then w.r.t. the source $J(y)$. Use the chain rule and the definition of $\langle\phi\rangle_J$, as well as the fact that $\frac{\delta J(x)}{\delta J(y)} = \delta^{(4)}(x-y)$.

c) All higher derivatives of the effective action are *one-particle irreducible* (1PI) n-point functions, i.e. the sum of all diagrams contributing to a n-point function that cannot be separated in two diagrams by cutting a propagator. Show this for the case $n = 3$.
Hints:

- Start from the full connected 3-point function, which you know is given by

$$\frac{1}{i^2}\frac{\delta^3 W}{\delta J(x_1)\delta J(x_2)\delta J(x_3)}$$

- Use the chain rule:

$$\frac{\delta}{\delta J(x_1)} = \int \mathrm{d}^4 y_1 \frac{\delta\langle\phi\rangle_J(y_1)}{\delta J(x_1)}\frac{\delta}{\delta\langle\phi\rangle_J(y_1)}$$

- Insert the result of part (b).

- To differentiate an inverse matrix:

$$\frac{\mathrm{d}M^{-1}}{\mathrm{d}\lambda} = -M^{-1}\frac{\mathrm{d}M}{\mathrm{d}\lambda}M^{-1}\,.$$

  This also holds for functional derivatives.

- What you end up with should correspond to the following picture: The LHS is what you started with, i.e. the full connected 3-point function. The RHS is the decomposition of this full amplitude into three full propagators connecting the remaining 1PI 3-point function.