Exercise 1) 6j-coefficients

We need to calculate:

$$\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 0\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix} .$$

a)

$$\begin{aligned}
\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 0\end{bmatrix} &= \sum_{l_1,l_2,l_3,l_{12},l_{23}} \begin{pmatrix}1 & 1 & 0\\ l_1 & l_2 & l_{12}\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ l_{12} & l_3 & l\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ l_2 & l_3 & l_{23}\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ l_1 & l_{23} & l\end{pmatrix}\\
&= \sum_{l_1,l_2,l_3} \begin{pmatrix}1 & 1 & 0\\ l_1 & l_2 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 0 & l_3 & l\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ l_2 & l_3 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ l_1 & 0 & l\end{pmatrix}\\
&\overset{\text{(i)}}{=} \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}\\
&\overset{\text{(ii)}}{=} \langle 11|\langle 10||00\rangle \cdot \langle 00|\langle 11||11\rangle \cdot \langle 10|\langle 11||00\rangle \cdot \langle 11|\langle 00||11\rangle\\
&\overset{\text{(iii)}}{=} \frac{1}{\sqrt{2}} \cdot 1 \cdot (-\frac{1}{\sqrt{2}}) \cdot 1 = -\frac{1}{2} ,
\end{aligned}$$

where we chose $l = 1$ (which we are allowed to since the 6j-coefficients are independent of l). In (i) we used that the weight of the vectors is additive. (ii) holds by the definition of the Clebsch-Gordan coefficients and (iii) follows from the calculation of these coefficients in the script.

Similarly we can calculate the other $6j$-coefficients:

$$\begin{aligned}
\begin{bmatrix}1 & 1 & 0\\ 1 & 1 & 2\end{bmatrix} &= \sum_{l_1,l_2,l_3,l_{12},l_{23}} \begin{pmatrix}1 & 1 & 0\\ l_1 & l_2 & l_{12}\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ l_{12} & l_3 & l\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ l_2 & l_3 & l_{23}\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ l_1 & l_{23} & l\end{pmatrix}\\
&= \sum_{l_1,l_2,l_{23}} \begin{pmatrix}1 & 1 & 0\\ l_1 & l_2 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ l_2 & 0 & l_{23}\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ l_1 & l_{23} & 0\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 0\end{pmatrix}\\
&+ \begin{pmatrix}1 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ 1 & 0 & 0\end{pmatrix}\\
&= \langle 10|\langle 11||00\rangle \cdot \langle 00|\langle 10||10\rangle \cdot \langle 11|\langle 10||21\rangle \cdot \langle 10|\langle 21||10\rangle\\
&+ \langle 11|\langle 10||00\rangle \cdot \langle 00|\langle 10||10\rangle \cdot \langle 10|\langle 10||20\rangle \cdot \langle 11|\langle 20||10\rangle\\
&= \frac{1}{\sqrt{2}} \cdot 1 \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{3}} + (-\frac{1}{\sqrt{2}}) \cdot 1 \cdot 1 \cdot (-\sqrt{\frac{2}{3}}) = \frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 0\end{bmatrix} &= \ldots = \sum_{l_1,l_2,l_3,l_{12}} \begin{pmatrix}1 & 1 & 2\\ l_1 & l_2 & l_{12}\end{pmatrix}\begin{pmatrix}2 & 1 & 1\\ l_{12} & l_3 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ l_2 & l_3 & 0\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ l_1 & 0 & 0\end{pmatrix}\\
&= \ldots = -\frac{\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix}1 & 1 & 2\\ 1 & 1 & 2\end{bmatrix} &= \sum_{l_1,l_2,l_3,l_{12},l_{23}} \begin{pmatrix}1 & 1 & 2\\ l_1 & l_2 & l_{12}\end{pmatrix}\begin{pmatrix}2 & 1 & 1\\ l_{12} & l_3 & 0\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ l_2 & l_3 & l_{23}\end{pmatrix}\begin{pmatrix}1 & 2 & 1\\ l_1 & l_{23} & 0\end{pmatrix}\\
&= \ldots = -\frac{1}{2} .
\end{aligned}$$

b) By definition

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \langle 1,1,1,0,1,0|_{\text{left}}|1,1,1,0,1,0\rangle_{\text{right}} \ ,$$

where we chose $l = 0$. Expressing the states in the computational basis we have

$$|1,1,1,0,1,0\rangle_{\text{left}} = \frac{1}{\sqrt{2}}|01-10\rangle \otimes |0\rangle \equiv \frac{1}{\sqrt{2}}|010-100\rangle$$
$$|1,1,1,0,1,0\rangle_{\text{right}} = |0\rangle \otimes \frac{1}{\sqrt{2}}|01-10\rangle \equiv \frac{1}{\sqrt{2}}|001-010\rangle \ ,$$

which let's us conclude that

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = -\frac{1}{2} \ .$$

Similarly we can calculate the other $6j$-coefficients:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \langle 1,1,1,0,1,0|_{\text{left}}|1,1,1,2,1,0\rangle_{\text{right}}$$
$$\overset{\text{(i)}}{=} \frac{1}{\sqrt{2}}\langle 010-100|(-\sqrt{\frac{2}{3}}|100\rangle + \frac{1}{\sqrt{6}}|010+001\rangle) = \frac{\sqrt{3}}{2} \ ,$$

where we used exercise 2b) of problem set 6 in (i).

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} = \langle 1,1,1,2,1,0|_{\text{left}}|1,1,1,0,1,0\rangle_{\text{right}}$$
$$\overset{\text{(i)}}{=} (-\sqrt{\frac{2}{3}}\langle 001| + \frac{1}{\sqrt{6}}\langle 100+010|)(\frac{1}{\sqrt{2}}|001-010\rangle) = -\frac{\sqrt{3}}{2} \ ,$$

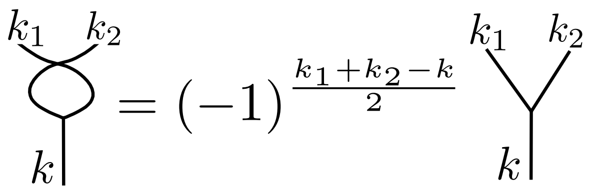
where we used exercise 2b) of problem set 6 in (i).

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \langle 1,1,1,2,1,0|_{\text{left}}|1,1,1,2,1,0\rangle_{\text{right}}$$
$$\overset{\text{(i)}}{=} (-\sqrt{\frac{2}{3}}\langle 001| + \frac{1}{\sqrt{6}}\langle 100+010|)(-\sqrt{\frac{2}{3}}|100\rangle + \frac{1}{\sqrt{6}}|010+001\rangle) = -\frac{1}{2} \ ,$$

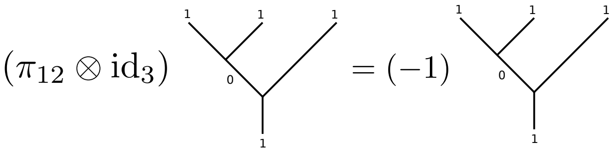
where we used exercise 2b) of problem set 6 in (i).

Exercise 2) Recoupling moves

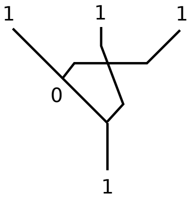
a) For $(\pi_{12} \otimes \text{id}_3)$ we use the following braiding rule:



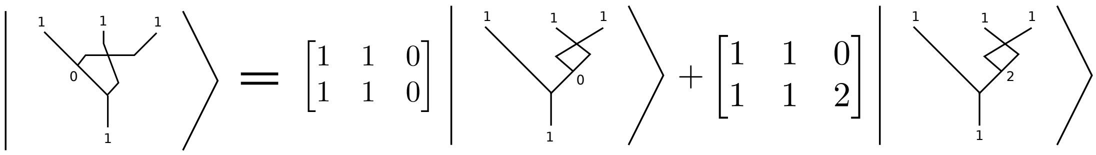
This gives us



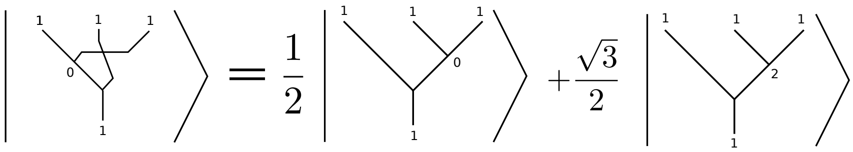
For $(\mathrm{id}_1 \otimes \pi_{23})$ we get a vector $|\beta\rangle$ which corresponds to



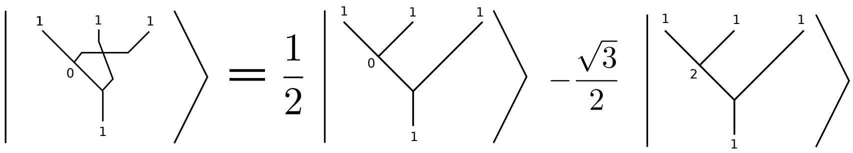
We can transform this from the left standard basis to the right standard basis:



Taking the $6j$-coefficients from exercise 1) and using the braiding rule from above, we get



Transforming the vectors on the RHS back to the left standard basis, we get after a short calculation analogue as above



b) We have $|\alpha\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes |1\rangle \equiv \frac{1}{\sqrt{2}}|011 - 101\rangle$ and according to exercise 2b) of problem set 6

$$(\pi_{12} \otimes \mathrm{id}_3) \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

hence $(\pi_{12} \otimes \mathrm{id}_3)|\alpha\rangle = -|\alpha\rangle$ as in a).

Furthermore we have according to exercise 2b) of problem set 6

$$(\mathrm{id}_1 \otimes \pi_{23}) \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} ,$$

and hence[1]

$$
\begin{aligned}
(\mathrm{id}_1 \otimes \pi_{23})|\alpha\rangle &= (\mathrm{id}_1 \otimes \pi_{23})\frac{1}{\sqrt{2}}|011 - 101\rangle \\
&= \frac{1}{2} \cdot (\frac{1}{\sqrt{2}}|011 - 101\rangle) + \frac{\sqrt{3}}{2} \cdot (\frac{1}{\sqrt{6}}|011 + 101\rangle - \sqrt{\frac{2}{3}}|110\rangle) \\
&= \frac{1}{\sqrt{2}}|011 - 110\rangle \ ,
\end{aligned}
$$

which is again the same as in a).

[1] Note that the signs of the matrix entries do not coincide with exercise 2a), because we chose the basis $\{\frac{1}{\sqrt{2}}|011 - 101\rangle, \sqrt{\frac{2}{3}}|110\rangle - \frac{1}{\sqrt{6}}|011 + 101\rangle\}$ and not the basis $\{\frac{1}{\sqrt{2}}|011 - 101\rangle, -\sqrt{\frac{2}{3}}|110\rangle + \frac{1}{\sqrt{6}}|011 + 101\rangle\}$ in exercise 2b) of problem set 6.