Exercise 1) S_4 **in Schur-Weyl Duality**

It is enough to specify the action of S_4 on $(\mathbb{C}^2)^{\otimes 4}$ on a generating set of S_4. We choose the set $\{\pi_{12} \otimes \mathrm{id}_{34}, \mathrm{id}_1 \otimes \pi_{23} \otimes \mathrm{id}_4, \mathrm{id}_{12} \otimes \pi_{34}\}$. The action of S_4 on $(\mathbb{C}^2)^{\otimes 4}$ can now be written as (in the standard basis):

$$\pi_{12} \otimes \mathrm{id}_{34} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 \,, \mathrm{id}_1 \otimes \pi_{23} \otimes \mathrm{id}_4 \mapsto \mathbb{1}_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes \mathbb{1}_2,$$

$$\mathrm{id}_{12} \otimes \pi_{34} \mapsto \mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ .$$

By the Schur transform we have

$$V_1^{\otimes 4} \cong (V_0 \otimes \mathbb{C}^2) \oplus (V_2 \otimes \mathbb{C}^3) \oplus (V_4 \otimes \mathbb{C}^1) \ .$$

So if we decompose the representation of S_4 on $(\mathbb{C}^2)^{\otimes 4}$ into a direct sum of irreducible representation, we know by the Schur-Weyl duality that a two-dimensional representation with multiplicity one, a three-dimensional representation with multiplicity three and a one-dimensional representation with multiplicity five appear.

The one-dimensional representation that appears is given by the trivial representation and the corresponding one-dimensional subspaces are given by

$$\begin{aligned}
|a\rangle &:= |0000\rangle \\
|b\rangle &:= |1111\rangle \\
|c\rangle &:= \frac{1}{2}|0001 + 0010 + 0100 + 1000\rangle \\
|d\rangle &:= \frac{1}{2}|0111 + 1011 + 1101 + 1110\rangle \\
|e\rangle &:= \frac{1}{\sqrt{6}}|0011 + 0101 + 0110 + 1001 + 1010 + 1100\rangle \ .
\end{aligned}$$

To specify the two-dimensional representation, we use

$$|f\rangle := |0, 0, - \ + \ -\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes \frac{1}{\sqrt{2}}|01 - 10\rangle \equiv \frac{1}{2}|0101 - 0110 - 1001 + 1010\rangle \ .$$

To get a second vector we note that the action of $SU(2)$ commutes with the action of S_4 and hence we just have to apply elements $\pi \in S_4$ to $|f\rangle$. The element $(\pi_{12} \otimes \mathrm{id}_{34})$ just gives a minus one, but for $(\mathrm{id}_1 \otimes \pi_{23} \otimes \mathrm{id}_4)$ we get

$$|g'\rangle := (\mathrm{id}_1 \otimes \pi_{23} \otimes \mathrm{id}_4)\frac{1}{2}|0101 - 0110 - 1001 + 1010\rangle = \frac{1}{2}|0011 - 0110 - 1001 + 1100\rangle \ .$$

This new vector $|g'\rangle$ is not orthogonal to $|f\rangle$, but using Gram-Schmidt orthonormalization we can get such a vector:

$$|g\rangle := \frac{1}{\sqrt{3}}|0011 + 1100\rangle - \frac{1}{2\sqrt{3}}|0101 + 0110 + 1001 + 1010\rangle \ .$$

The space spanned by $\{|f\rangle, |g\rangle\}$ now gives the space of the two-dimensional representation. The action in the basis $\{|f\rangle, |g\rangle\}$ can be calculated to

$$\pi_{12} \otimes \mathrm{id}_{34} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathrm{id}_1 \otimes \pi_{23} \otimes \mathrm{id}_4 \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ \mathrm{id}_{12} \otimes \pi_{34} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} .$$

To specify the first three-dimensional representation, we use

$$|h\rangle := |2, 0, - \ + \ +\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes |2, 0\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes |00\rangle \equiv \frac{1}{\sqrt{2}}|0010 - 1000\rangle .$$

Similarly as before we can get

$$|i\rangle := \sqrt{\frac{2}{3}}|0010\rangle - \frac{1}{\sqrt{6}}|1000 + 0100\rangle ,$$

as well as

$$|j\rangle := \frac{1}{2\sqrt{3}}|0010 + 0100 + 1000\rangle - \frac{\sqrt{3}}{2}|0001\rangle .$$

The space spanned by $\{|h\rangle, |i\rangle, |j\rangle\}$ now gives the space of the first three-dimensional representation. The action in the basis $\{|h\rangle, |i\rangle, |j\rangle\}$ can be calculated to

$$\pi_{12}\otimes\mathrm{id}_{34} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathrm{id}_1\otimes\pi_{23}\otimes\mathrm{id}_4 \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathrm{id}_{12}\otimes\pi_{34} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} .$$

To specify the second three-dimensional representation, we use

$$\begin{aligned} |k\rangle &:= |2, 1, - \ + \ +\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes |2, 1\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes \frac{1}{\sqrt{2}}|01 + 10\rangle \\ &\equiv \frac{1}{2}|0101 + 0110 - 1010 - 1001\rangle . \end{aligned}$$

Similarly as before we can get

$$|l\rangle := \frac{1}{\sqrt{3}}|0011 - 1100\rangle + \frac{1}{2\sqrt{3}}|0110 - 0101 - 1001 + 1010\rangle ,$$

as well as

$$|m\rangle := \frac{1}{\sqrt{3}}|0011 - 1100\rangle - \frac{1}{2\sqrt{3}}|0110 - 0101 - 1001 + 1010\rangle .$$

The space spanned by $\{|k\rangle, |l\rangle, |m\rangle\}$ now gives the space of the second three-dimensional representation. The action in the basis $\{|k\rangle, |l\rangle, |m\rangle\}$ can be calculated to

$$\pi_{12}\otimes\mathrm{id}_{34} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathrm{id}_1\otimes\pi_{23}\otimes\mathrm{id}_4 \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \mathrm{id}_{12}\otimes\pi_{34} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} .$$

To specify the third three-dimensional representation, we use

$$|n\rangle := |2, 2, - \ + \ +\rangle = \frac{1}{\sqrt{2}}|01 - 10\rangle \otimes |11\rangle \equiv \frac{1}{\sqrt{2}}|0111 - 1011\rangle .$$

Similarly as before we can get

$$|o\rangle := \frac{1}{\sqrt{6}}|0111 + 1011\rangle - \sqrt{\frac{2}{3}}|1101\rangle \ ,$$

as well as

$$|p\rangle := \frac{1}{2\sqrt{3}}|0111 + 1011 + 1101\rangle - \frac{\sqrt{3}}{2}|1110\rangle \ .$$

The space spanned by $\{|n\rangle, |o\rangle, |p\rangle\}$ now gives the space of the third three-dimensional representation. The action in the basis $\{|n\rangle, |o\rangle, |p\rangle\}$ can be calculated to

$$\pi_{12}\otimes\mathrm{id}_{34} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} ,\ \mathrm{id}_1\otimes\pi_{23}\otimes\mathrm{id}_4 \mapsto \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} ,\ \mathrm{id}_{12}\otimes\pi_{34} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \end{pmatrix} .$$

Exercise 2) The Sign Representation of S_n

Let $\{|j_1\rangle\}_{j_1=1,2,\ldots,n}$ be an orthonormal basis of $\mathbb{C}^n$ and $\{|j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_n\rangle\}_{j_k\in\{1,2,\ldots,n\}}$ be the corresponding tensor product basis of $(\mathbb{C}^n)^{\otimes n}$.

The natural action of S_n on $(\mathbb{C}^n)^{\otimes n}$ is given by $S_n \ni \pi \mapsto P(\pi)$ with

$$P(\pi)(|j_1\rangle \otimes |j_2\rangle \otimes \ldots \otimes |j_n\rangle) = |j_{\pi^{-1}(1)}\rangle \otimes |j_{\pi^{-1}(2)}\rangle \otimes \ldots \otimes |j_{\pi^{-1}(n)}\rangle \ .$$

The decomposition of this representation into irreducible representations gives that the one-dimensional sign representation

$$S_n \ni \pi \mapsto \mathrm{sign}(\pi)$$

appears with multiplicity one (e.g. this can be done using Schur-Weyl duality).

The corresponding one dimensional subspace is spanned by

$$|\alpha^n\rangle = \frac{1}{\sqrt{n!}} \sum_{\pi\in S_n} \mathrm{sign}(\pi)|j_{\pi(1)}\rangle \otimes |j_{\pi(2)}\rangle \otimes \ldots \otimes |j_{\pi(n)}\rangle \ .$$

This is because we have for any $\tilde{\pi} \in S_n$ that

$$\begin{aligned} P(\tilde{\pi})|\alpha^n\rangle &= \frac{1}{\sqrt{n!}} \sum_{\pi\in S_n} \mathrm{sign}(\pi)|j_{\tilde{\pi}^{-1}(\pi(1))}\rangle \otimes |j_{\tilde{\pi}^{-1}(\pi(2))}\rangle \otimes \ldots \otimes |j_{\tilde{\pi}^{-1}(\pi(n))}\rangle \\ &= \frac{1}{\sqrt{n!}} \sum_{\hat{\pi}\in S_n} \mathrm{sign}(\tilde{\pi}\hat{\pi})|j_{\hat{\pi}(1)}\rangle \otimes |j_{\hat{\pi}(2)}\rangle \otimes \ldots \otimes |j_{\hat{\pi}(n)}\rangle \\ &= \mathrm{sign}(\tilde{\pi})\frac{1}{\sqrt{n!}} \sum_{\hat{\pi}\in S_n} \mathrm{sign}(\hat{\pi})|j_{\hat{\pi}(1)}\rangle \otimes |j_{\hat{\pi}(2)}\rangle \otimes \ldots \otimes |j_{\hat{\pi}(n)}\rangle \\ &= \mathrm{sign}(\tilde{\pi})|\alpha^n\rangle \ , \end{aligned}$$

where we used $\hat{\pi} := \tilde{\pi}^{-1}\pi$.