Exercise 1) Tensor Product Representations

The n-fold tensor product representation is given by

$$\begin{aligned} V_1^{\otimes n} : SU(2) &\to SU(2^n) \\ g &\mapsto g^{\otimes n} \ . \end{aligned}$$

By exercise 2 of problem set 3 the corresponding Lie algebra representation $v_1^{\otimes n} : su(2) \to su(2^n)$ is given by

$$\begin{aligned} a \mapsto & -i\frac{d}{dt}V_1^{\otimes n}(\exp(ita))|_{t=0} = -i\frac{d}{dt}(\exp(ita))^{\otimes n}|_{t=0} \\ &= a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \mathbb{1} \otimes a \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} + \ldots + \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes a \ . \end{aligned}$$

Exercise 2) Schur Transform

Because the weight of the vectors is additive we can get

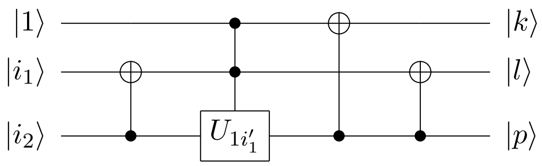
$$2(i_1 + i_2 + \ldots + i_n) - n = 2l - k \ ,$$

where the LHS is the weight of $|i_1\rangle \otimes |i_2\rangle \otimes \ldots \otimes |i_n\rangle$ and the RHS is the weight of $|k, l\rangle$. This gives

$$l = i_1 + i_2 + \ldots + i_n + \frac{k-n}{2} \ . \tag{1}$$

Therefore the output depends on k. If k is known the measurement of l will yield the corresponding value in (1). Otherwise all values in (1) can appear (and each with a certain proability).

For $n = 2$ the Schur transform is given by the following circuit:



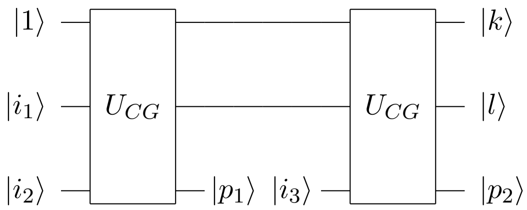
where

$$U_{1i_1'} = \begin{pmatrix} \sqrt{\frac{i_1'}{2}} & \sqrt{1-\frac{i_1'}{2}} \\ -\sqrt{1-\frac{i_1'}{2}} & \sqrt{\frac{i_1'}{2}} \end{pmatrix} \ .$$

The input $|i_1 i_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ can now be calculated to give the output $|k, l\rangle = |0, 0\rangle$.

Alike the input $|i_1 i_2\rangle = |00\rangle$ gives the output $|k, l\rangle = |2, 0\rangle$, $|i_1 i_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ gives $|k, l\rangle = |2, 1\rangle$ and $|i_1 i_2\rangle = |1, 1\rangle$ gives $|k, l\rangle = |2, 2\rangle$.

For $n = 3$ the circuit for the Schur transform is given by



It is now easy to see that the input $|i_1 i_2 i_3\rangle = |111\rangle$ is needed to get the output $|k, l\rangle = |3, 3\rangle$.

Exercise 3) Asymptotic Behavior of m_k^n

By Stirling's approximation we have for $n \to \infty$ that

$$m_k^n = \binom{n}{\frac{n-k}{2}} \cdot (\frac{2k+2}{n+k+2}) = \frac{n!}{(\frac{n-k}{2})!(\frac{n+k}{2})!} \cdot \frac{2k+2}{n+k+2}$$
$$\approx \frac{\sqrt{2\pi n} \cdot (\frac{n}{e})^n}{\sqrt{\pi(n-k)}\sqrt{\pi(n+k)} \cdot (\frac{n-k}{2e})^{\frac{n-k}{2}} \cdot (\frac{n+k}{2e})^{\frac{n+k}{2}}} \cdot \frac{2k+2}{n+k+2} \ .$$

Hence we get for $k = n \cdot c$ that

$$m_k^n \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n(1-c)n(1+c)}} \cdot \frac{(\frac{n}{e})^n}{(\frac{n}{e} \cdot \frac{1-c}{2})^{\frac{n(1-c)}{2}} \cdot (\frac{n}{e} \cdot \frac{1+c}{2})^{\frac{n(1+c)}{2}}} \cdot \frac{2cn+2}{n(1+c)+2}$$
$$= \sqrt{\frac{2}{\pi(1+c^2)}} \cdot \frac{1}{n} \cdot (\frac{1-c}{2})^{-\frac{n(1-c)}{2}} \cdot (\frac{1+c}{2})^{-\frac{n(1+c)}{2}} \cdot \frac{2cn+2}{n(1+c)+2} \ .$$

Finally we get

$$\lim_{n\to\infty} \frac{1}{n} \log(m_k^n) = -\frac{(1-c)}{2} \cdot \log(\frac{1-c}{2}) - \frac{(1+c)}{2} \cdot \log(\frac{1+c}{2}) = h(\frac{1-c}{2}) \ .$$