Exercise 1) Permutation Group S_n

Show that every $\pi \in S_n$ can be written as a product of at most $O(n^2)$ transpositions of pairs of neighbors.

Exercise 2) Two-Qubit Gate

Provide a decomposition of

$$U = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \frac{1+i}{\sqrt{2}} & -\frac{1+i}{2} & -\frac{1+i}{2} \\ 0 & \frac{1-i}{\sqrt{2}} & \frac{3-i}{2} & -\frac{1+i}{2} \\ 1 & -1 & \frac{1+i}{\sqrt{2}} & -\frac{1-i}{\sqrt{2}} \\ 1 & -i & 0 & \sqrt{2} \end{pmatrix}$$

into two-level unitaries (i.e. unitaries that act nontrivially only on two computational basis states).

Exercise 3) Two-Level Unitary

Obtain a quantum circuit composed of single qubit unitaries and CNOT gates that implements the two-level unitary

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & d \end{pmatrix}$$

where $\tilde{U} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an arbitrary (2×2)-unitary matrix.