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ETH Zurich, HS 2010

Exercise 1) Bloch Vector

Show that any quantum state ρ on $\mathcal{H} \cong \mathbb{C}^2$ takes the form $\rho = \frac{1}{2}(\mathbf{1} + \vec{v} \cdot \vec{\sigma})$ where $\vec{\sigma}$ denotes the vector of Pauli matrices and $\vec{v} \in \mathbb{R}^3$ with $|\vec{v}|^2 \leq 1$ ($\vec{v}$ is called Bloch vector). Represent the states $|0\rangle\langle 0|$, $|1\rangle\langle 1|$, $|+\rangle\langle +|$, $|-\rangle\langle -|$, where $|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, and $\mathbf{1}/2$ graphically.

Exercise 2) Average State

Let $\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i|$ be a decomposition of ρ into pure states $|\phi_i\rangle$ ($p_i \geq 0$, $\sum_i p_i = 1$). Note that the number of terms in the sum is not limited by the dimension of the system. For which states ρ is this decomposition unique when

1. the $|\phi_i\rangle$ are orthonormal?
2. they are not necessarily orthonormal?

Construct a few examples in dimension two in order to get a feeling for the situation.

Exercise 3) Partial Trace

The partial trace is an important concept in the quantum mechanical treatment of multi-partite systems and it is the natural generalization of the concept of marginal distributions in classical probability theory.
Let $\rho_{AB} = \sum_{klmn} \rho_{klmn} |k\rangle\langle l|_A \otimes |m\rangle\langle n|_B$ be a density operator on the bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\{|k\rangle_A\}$ and $\{|l\rangle_B\}$ are orthonormal bases for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The partial trace is defined by $\rho_A = \mathrm{tr}_B(\rho_{AB}) = \sum_{kl} |k\rangle\langle l|_A \mathrm{tr}[(|l\rangle\langle k|_A \otimes \mathbf{1}_B)\rho_{AB}]$. Show that $\rho_A = \sum_{klm} \rho_{klmm} |k\rangle\langle l|_A$ and prove that it is a valid density operator by confirming the following properties:

1. Hermiticity: $\rho_A = \rho_A^\dagger$
2. Positivity: $\rho_A \geq 0$
3. Normalization: $\mathrm{tr}\rho_A = 1$.

Calculate the reduced density matrix ρ_A of $\rho_{AB} = |\phi\rangle\langle\phi|$, where $|\phi\rangle_{AB} = \sqrt{r}|00\rangle_{AB} + \sqrt{1-r}|11\rangle_{AB}$ and $0 \leq r \leq 1$. How is r related to the Bloch vector?

Let $P_{XY}(x, y)$ with $x, y \in \{1, \dots d\}$ be a probability distribution, i.e. $P_{XY}(x, y) \geq 0$ for all x, y and $\sum_{x,y} P_{XY}(x, y) = 1$. Compute the marginal distributions P_X and P_Y and verify that they are probability distributions. How can we represent P_{XY} in form of a quantum state? Calculate the partial traces of P_{XY} in its quantum representation.