IX. SPONTANEOUS SUPERSYMMETRY BREAKING

Up to now we have seen how to construct supersymmetric theories, however in Nature we do not observe the degeneracy of masses between bosons and fermions predicted by supersymmetry. This means that, if supersymmetry is really a symmetry of Nature, it must be broken.
There are two common ways to break supersymmetry:

- Explicit breaking, obtained by introducing (soft) susy breaking terms in the action;
- Spontaneous breaking, due to the existence of a non-susy-invariant vacuum.

Often explicit breaking by soft terms is a way to parametrize in a low energy effective theory the effects of a spontaneous supersymmetry breaking at high energy. An example of this procedure is the parametrization of susy breaking in the MSSM which is done by introducing soft susy-breaking terms in the theory.
In this section we will discuss the mechanism of spontaneous susy breaking.

Supersymmetry breaking and vacuum energy

We already saw that in susy theories energy is non-negative. This means that, in particular, the vacuum has non-negative energy. Let's use the susy algebra to write

$$\langle 0|P_0|0\rangle = \frac{1}{4}\|Q_\alpha|0\rangle\|^2 + \frac{1}{4}\|\bar{Q}_{\dot\alpha}|0\rangle\|^2 \geq 0,$$

where $|0\rangle$ is the vacuum. If the vacuum is invariant under susy transformations (that is susy is not spontaneously broken) we have

$$\underline{Q_\alpha|0\rangle = \bar{Q}_{\dot\alpha}|0\rangle = 0 \quad \Rightarrow \quad \langle 0|P_0|0\rangle = 0 .}$$

On the other hand, if the vacuum is not invariant (that is susy is spontaneously broken) we get

$$\underline{Q_\alpha|0\rangle \neq 0 \quad \Rightarrow \quad \langle 0|P_0|0\rangle > 0 .}$$

The above statements can also be reversed, that is we get

$$Q_\alpha|0\rangle = \bar{Q}_{\dot\alpha}|0\rangle = 0 \quad \Longleftrightarrow \quad \langle 0|P_0|0\rangle = 0 ,$$

and

$$Q_\alpha|0\rangle \neq 0 \quad \Longleftrightarrow \quad \langle 0|P_0|0\rangle > 0 .$$

This gives an important restatement of the condition for spontaneous supersymmetry breaking:
<u>supersymmetry is spontaneously broken if and only if the vacuum energy is positive.</u>

· Vacua in susy theories.

Another criterion of spontaneous supersymmetry breaking can be given in terms of vacuum expectation values of the auxiliary fields. Let's make a step backwards and discuss first of all the properties of the vacuum and how it can be found.

A first important property comes from Poincaré invariance. Of course we do not want a spontaneous breaking of the Poincaré invariance, so we must always require that the vacuum configuration respects this symmetry. In other words, we need to impose that all the fields must have constant values on Minkowski space (that is their spacetime derivatives must vanish) and all the fields which are non-scalars must have zero expectation value. Only scalar fields z^i can have a non-vanishing VEV : $\langle z^i \rangle$.

The vacuum configuration is the one which minimizes the value of the euclidean action (or equivalently minimizes the energy), this is equivalent to the requirement that the scalar potential V is minimized by the vacuum configuration. Thus we have for a vacuum

$$\langle v^a_\mu \rangle = \langle \lambda^a \rangle = \langle \psi^i \rangle = \partial_\mu \langle z^i \rangle = 0 \quad , \quad V(\langle z^i \rangle, \langle z^{i+} \rangle) = \text{minimum}.$$

The minimum may be the global minimum of V, in which case one has the true vacuum, or it may be a local minimum, in which case one has a false (metastable) vacuum. In any case, for a true or false vacuum one has

$$\frac{\partial V}{\partial z^i}(\langle z^i \rangle, \langle z^{i+} \rangle) = \frac{\partial V}{\partial z^{i+}}(\langle z^i \rangle, \langle z^{i+} \rangle) = 0.$$

This shows that the vacuum is a solution of the equations of motion.

In a supersymmetric theory the scalar potential is given by

$$V(z, z^+) = F_i^+ F_i + \frac{1}{2} D^a D^a,$$

where

$$F_i^+ = \frac{\partial W(z)}{\partial z^i}$$

and

$$D^a = -z^+ T^a z - \xi^a .$$

The potential is non-negative, so it will certainly be at a global minimum, namely $V = 0$, if

$$F^i(\langle z^+ \rangle) = D^a(\langle z \rangle, \langle z^+ \rangle) = 0. \qquad (*)$$

In this case supersymmetry is unbroken, given that the vacuum energy is zero.

Notice that, however, the system of equations (*) does not necessarily have a solution. We can have two cases:

i) If equations (*) have a solution, this solution is a global minimum with $V=0$, hence supersymmetry is unbroken. Note that there can be many solutions of the (*) equations, and all of them correspond to degenerate susy-invariant vacua.
In addition there could be some false vacua corresponding to local minima of the potential with $V \neq 0$. These are metastable vacua with broken supersymmetry.

ii) If equations (*) have no solution, the scalar potential can never vanish and its minimum is strictly positive: $V>0$. This means that the vacuum is not susy-invariant and supersymmetry is necessarily spontaneously broken.

As we have seen, spontaneous susy breaking is related directly to the values of the auxiliary fields F^i and D^a on the vacuum. We have that susy is broken if and only if some of the F^i and/or the D^a has non-zero value on the vacuum.
There is also another way to see that susy breaking is related to F^i and D^a. By looking at the susy transformations of the chiral superfields components we get

$$\begin{cases} \delta\langle z^i\rangle = \sqrt{2}\,\varepsilon\langle\psi^i\rangle \\ \delta\langle\psi^i\rangle = \sqrt{2}\, i\partial_\mu\langle z^i\rangle\sigma^\mu\bar{\varepsilon} - \sqrt{2}\langle F^i\rangle\varepsilon \\ \delta\langle F^i\rangle = \sqrt{2}\, i\,\partial_\mu\langle\psi^i\rangle\sigma^\mu\bar{\varepsilon} \end{cases}$$

which, taking into account the conditions given by the unbroken Poincaré group, become

$$\delta\langle z^i\rangle = 0$$
$$0 = \delta\langle\psi^i\rangle = -\sqrt{2}\langle F^i\rangle\varepsilon$$
$$\delta\langle F^i\rangle = 0$$

The two conditions on $\delta\langle\psi^i\rangle$ come, on one side, from the susy transformations and, on the other side, from the fact that $\langle\psi^i\rangle$ must vanish (as well as its variation) on a Poincaré-invariant vacuum.
The above conditions, which encode the requirements to have a susy-invariant vacuum, can be consistent only if $\langle F^i\rangle = \bar{F}^i(\langle z^{i+}\rangle) = 0$. Differently, if $\langle F^i\rangle \neq 0$ for some i, the vacuum is not susy-invariant and susy is spontaneously broken.

We can find a similar argument for D^a. Under susy transformations

$$\begin{cases} \delta f^a_{\mu\nu} = \varepsilon(\sigma_\mu\partial_\nu - \sigma_\nu\partial_\mu)\bar{\lambda}^a - \bar{\varepsilon}(\bar{\sigma}_\mu\partial_\nu - \bar{\sigma}_\nu\partial_\mu)\lambda^a \\ \delta\lambda^a = i\varepsilon D^a - \sigma^{\mu\nu}\varepsilon f^a_{\mu\nu} \\ \delta D^a = -\partial_\mu(\lambda^a\sigma^\mu\bar{\varepsilon} + \varepsilon\sigma^\mu\bar{\lambda}^a) \end{cases}$$

Computing these relations on the vacuum we get the conditions for susy (and Poincaré) invariance

$$\delta \langle f^a_{\mu\nu} \rangle = 0$$

$$0 = \delta \langle \lambda^a \rangle = i\varepsilon \langle D^a \rangle$$

$$\delta \langle D^a \rangle = 0$$

Again we see that the auxiliary field D^a is related to the breaking of susy. If $\langle D^a \rangle \neq 0$ on the vacuum, then the vacuum is not susy-invariant and susy is spontaneously broken.

· The Goldstone theorem for susy.

Goldstone's theorem states that, whenever a continuous global symmetry is spontaneously broken, there is a massless mode in the spectrum, i.e. a massless particle. The quantum numbers carried by the Goldstone particle are related to the broken symmetry.

Similarly if supersymmetry is spontaneously broken there is a massless spin one-half particle, i.e. a massless fermion, usually called goldstino.

Now we will explicitly prove that in the presence of spontaneously broken susy we get a massless fermion. A vacuum which breaks susy is such that

$$\begin{cases} \dfrac{\partial V}{\partial z^i}\left(\langle z^j \rangle, \langle z^{j+} \rangle\right) = 0 \\ \langle F^i \rangle \neq 0 \quad \text{and/or} \quad \langle D^a \rangle \neq 0 \end{cases}$$

From the explicit form of V we get

$$\frac{\partial V}{\partial z^i} = F^j \frac{\partial^2 W}{\partial z^i \partial z^j} - D^a z_j^+ (T^a)^j{}_i$$

and this vanishes on the vacuum. The statement that the superpotential is gauge-invariant can be written as

$$0 = \delta^{(a)}_{gauge} W = \frac{\partial W}{\partial z^i} \delta^{(a)}_{gauge} z^i = F_i^+ (T^a)^i{}_j z^j .$$

We can now combine the above equations into the matrix equation

$$M = \begin{pmatrix} \dfrac{\partial^2 W}{\partial z^i \partial z^j} & -\langle z_\ell^+ \rangle (T^a)^\ell{}_i \\ -\langle z_\ell^+ \rangle (T^b)^\ell{}_j & 0 \end{pmatrix} , \qquad M \begin{pmatrix} \langle F^j \rangle \\ \langle D^a \rangle \end{pmatrix} = 0 ,$$

which states that the matrix M has a zero eigenvalue.

But the matrix M is exactly the fermion mass matrix. We already discussed the form of the gauge Lagrangian with chiral superfields and we found that the fermion mass term (in the presence of a VEV for the scalar fields) is

$$\left(i\sqrt{2}\langle z_j^+\rangle (T^a)^j{}_i \lambda^a \psi^i - \frac{1}{2}\left\langle \frac{\partial^2 W}{\partial z^i \partial z^j}\right\rangle \psi^i\psi^j\right) + h.c.$$

$$= -\frac{1}{2}\left(\psi^i, \sqrt{2}\, i\lambda^b\right) M \begin{pmatrix} \psi^j \\ \sqrt{2}\, i\lambda^a \end{pmatrix} + h.c.$$

This mass matrix has a zero eigenvalue and this means that there is a massless fermion: this is the Goldstone fermion or Goldstino.

Mechanisms of supersymmetry breaking

Now we are ready to analyse the possible mechanisms which can generate a spontaneous breaking of rigid $N=1$ supersymmetry. In the following we will start by analyzing the case of a theory with only chiral superfields and then we will consider the issue of susy breaking in gauge theories.

· Supersymmetry breaking in theories with only chiral superfields

As we have seen before, susy is spontaneously broken if the scalar potential of the theory is always strictly positive. In the case of a theory with only chiral superfields the scalar potential comes only from the superpotential $W(\Phi)$, that is it is generated by an F-term. In general we have

$$V = \sum_i F_i^+ F_i = \sum_i \left|\frac{\partial W}{\partial z^i}\right|^2$$

and susy is spontaneously broken if and only if the system of equations

$$F_i^+ = \frac{\partial W}{\partial z^i} = 0$$

admits no solution. Notice that there are as many independent variables as there are equations to satisfy, so we generally expect a solution to exist. In order for supersymmetry to be broken in these theories, it is necessary to impose restrictions on the form of the superpotential.

In the simplest possible case, the theory of one chiral superfield, we have a complex equation

$$F^+ = \frac{\partial W}{\partial z} = 0 .$$

In general $\frac{\partial W}{\partial z}$ will be a complex polynomial in z, so it always has at least one zeros in the complex plane and supersymmetry can not be broken.

(NOTE. An exception to the above statement is obviously given by the trivial case in which $\frac{\partial W}{\partial z} = \text{const} \neq 0$, or, in other words $W(\Phi) = \lambda \Phi$. However in this case it is easy to see that the theory is just a free field theory of a scalar and a fermion field without mass terms or interactions, so we can neglect this uninteresting case.)

To get a theory with susy breaking we thus need to consider more than one chiral superfield.

The most interesting cases, clearly, are the ones which involve a renormalizable theory, so we will concentrate on this case in the following discussion. For a renormalizable theory we have

$$W(\Phi_i) = a_i \Phi_i + m_{ij} \Phi_i \Phi_j + g_{ijk} \Phi_i \Phi_j \Phi_k ,$$

thus the set of equations which determine the breaking of susy are

$$F_i^+ = \frac{\partial W}{\partial z^i} = a_i + m_{ij} z^j + g_{ijk} z^j z^k = 0 .$$

We can notice that, if $a_i = 0$ for each i, the system of equations has a trivial solution $\langle z^i \rangle = 0$ for each i, so susy is unbroken. Thus a necessary condition to have susy breaking is that some of the a_i should be non-zero. This however is <u>not</u> a sufficient condition for susy breaking as can be easily checked.

Notice that we can redefine the fields in our theory by shifting the fields by a constant: $\Phi_i \to \Phi_i + b_i$. This transformation changes only the z^i scalar field component of Φ_i in a way which respects susy transformations (the variation of z^i is not changed by adding a constant, $\delta(z^i + b^i) = \delta z^i$, and z^i appears only in the variation of ψ with a spacetime derivative, $\delta\psi^i = \sqrt{2}\, i\sigma^\mu \bar{\varepsilon}\, \partial_\mu z^i - \sqrt{2}\, \varepsilon F^i$), moreover, in a renormalizable theory also the kinetic term is unchanged by the shift. With this transformation we get a new superpotential with parameters

$$\begin{cases} a'_i = a_i + m_{ij} b_j + g_{ijk} b_j b_k \\ m'_{ij} = m_{ij} + g_{ijk} b_k \\ g'_{ijk} = g_{ijk} \end{cases}$$

Notice that the condition to have unbroken susy is equivalent to the possibility of setting all the $a'_i = 0$ by a shift in the fields. The reason is simple: by the shift we can move the minimum of the potential to the origin, $\langle z^i \rangle = 0 \ \forall i$, if the minimum does not break susy, then we get a new potential which vanishes for $\langle z^i \rangle = 0$, so all the a'_i must be zero.

It can be shown that, in renormalizable theories with only two chiral superfields, susy can not be spontaneously broken, or, in other words the potential has at least one zero. The simplest models which exhibit spontaneous susy breaking have three chiral superfields and have been proposed by O'Raifeartaigh. In the following we will describe a generalization of the class of models due to O'Raifeartaigh. We consider a superpotential which is a linear combination of a set Y_i of chiral superfields, with coefficients given by functions $h_i(X)$ of a second set of chiral superfields X_n:

$$W(X,Y) = \sum_i Y_i \, h_i(X).$$

The conditions for susy to be unbroken are

$$\begin{cases} 0 = \dfrac{\partial W(x,y)}{\partial y_i} = h_i(x) \\[2ex] 0 = \dfrac{\partial W(x,y)}{\partial x_n} = \sum_i y_i \dfrac{\partial h_i(x)}{\partial x_n} \end{cases}$$

The second set of equations can always be solved by taking $y_i = 0$, with no effect on the problem of solving the first set of equations. But, if the number of X_n superfields is smaller than the number of Y_i superfields, then the first set of equations imposes more conditions on the x_n than the number of free variables, so, without fine-tuning, a solution does not exist and supersymmetry is broken.

Notice that the initial assumption of the form of the superpotential can be naturally obtained (without fine-tuning) by assuming a suitable R-symmetry. We can obtain the wanted structure for the superpotential by requiring R-invariance, with R-charges $+2$ for the Y_i and 0 for the X_n (remember that the superpotential must contain terms of R-charge $+2$ to get an R-invariant Lagrangian).

The scalar potential of this class of models is

$$V(x,y) = \sum_i |h_i(x)|^2 + \sum_n \left| \sum_i y_i \frac{\partial h_i(x)}{\partial x_n} \right|^2 .$$

This potential is always minimized by choosing the x_n to minimize the first term, while the second term can always be minimized by choosing $y_i = 0$. These models, however, have a peculiar feature: there are always directions in the space of fields in which the minimum of the potential is flat. If $x_n = \langle x_n \rangle$ are a set of values which minimize the first term, the second term vanishes not only for $y_i = 0$, but also for any vector y_i in a direction orthogonal to all the vectors $(v^n)_i = (\partial h_i(x)/\partial x_n)_{x=\langle x \rangle}$. If there are N_X superfields X_n and N_Y superfields Y_i with $N_Y > N_X$, then there will be at least $N_Y - N_X$ flat directions. Notice that for any non-vanishing value of the y_i R-symmetry is broken.

The simplest example of this class of models is provided by the case with one X and two Y superfields. Renormalizability requires the coefficients $h_i(X)$ to be quadratic functions of X, and, by taking suitable linear combinations of the Y_i and shifting and rescaling X, we can choose these functions as

$$h_1(X) = X - a \quad , \qquad h_2(X) = X^2,$$

with an arbitrary constant a. Susy is clearly broken unless $a=0$. The potential is

$$V(x,y) = |x|^4 + |x-a|^2 + |y_1 + 2xy_2|^2 .$$

The first two terms have a unique global minimum x_0. The flat direction is the one for which $y_1 + 2x_0y_2 = 0$. For $a=0$ we get unbroken susy and the minima have $x_0=0$, $y_1=0$ and y_2 arbitrary.

NOTE. The coordinates which parametrize the degenerate vacua are called "moduli". The set of degenerate vacua is the "moduli space".

· Supersymmetry breaking in gauge theories

Let's now discuss what happens in supersymmetric gauge theories. In this case the scalar potential is

$$V(z,z^+) = F_i^+ F_i + \frac{1}{2} D^a D^a$$

where

$$F_i^+ = \frac{\partial W(z)}{\partial z^i}$$

and

$$D^a = -z^+ T^a z - \xi^a ,$$

where the ξ^a can be non-zero only for the Abelian subgroups of the gauge group. Naively one could think that supersymmetry breaking is very common given that there are more equations than variables. However, for a gauge group of dimensionality D the superpotential $W(\Phi)$ is subject to the D constraints

$$\sum_m \frac{\partial W(z)}{\partial z^m} (T^a z)_m = 0$$

for all a. Hence if there are n independent chiral superfields, then the number of independent conditions is exactly n, equal to the number of variables. With the number of conditions equal to the number of free variables, it is likely to find solutions for generic superpotentials, thus supersymmetry is usually not broken.

It is easy to see that, if all the Fayet-Iliopoulos constants ξ^a vanish, if there exists a solution of the conditions $F_i^+ = 0$, then there is another configuration which satisfies all the conditions for susy to be unbroken. This means that in this case the mechanism of susy breaking is similar to the one for theories with only chiral fields, that is susy breaking can be induced only by non-vanishing F_i terms and it can not happen a breaking induced only by the D^a terms. Let's show this. We notice that the superpotential $W(\Phi)$ does not involve $\bar{\Phi}$, so it is invariant not only under ordinary gauge transformations $\Phi \to \exp(i \sum_A T^A \lambda^A)\Phi$ with λ^A arbitrary real numbers, but also transformations with λ^A arbitrary complex numbers. Under all these transformations the F_i^+ (and F_i) terms transform linearly, so if z_0^i satisfies $F_i^+ = 0$, then so does $z^\lambda \equiv \exp(i \sum_A T^A \lambda^A) z_0$. On the other hand, the scalar product $z^+ z$ is not invariant under the transformations with λ^A complex, but $z^{+\lambda} z^\lambda$ remains real and positive for complex λ^A, so it is bounded from below and, therefore, has a minimum. For $\xi^a = 0$, the condition that $z^{+\lambda} z^\lambda$ is at a minimum is just

$$z^{\lambda +} T^a z^\lambda = 0 \ ,$$

which tells us that the D^a vanish. Thus in the absence of Fayet-Iliopoulos terms the question of susy breaking is entirely determined by the superpotential.

Now we can present some explicit models which show different patterns of symmetry breaking. We will see that the breaking of supersymmetry and the breaking of the gauge symmetry are independent phenomena and we can build models in which both symmetries or just one of them is spontaneously broken.

Notice a connection between the auxiliary fields D^a and the breaking of gauge symmetry. The D^a auxiliary components are not gauge invariant (unless they correspond to an Abelian subgroup), so if D^a gets a non-vanishing VEV one has spontaneous gauge breaking. Notice that this is not a necessary condition for gauge symmetry breaking, one can have gauge breaking also if susy is unbroken (and thus the D^a have zero VEV), in this case it is the VEV of some scalar field z^i which breaks the gauge invariance.

· Fayet-Iliopoulos supersymmetry breaking

The existence of Fayet-Iliopoulos terms for $U(1)$ gauge subgroup gives another mechanism to break supersymmetry. The simplest case is a theory with a $U(1)$ gauge group.

(NOTE. In order to avoid $U(1)$-$U(1)$-$U(1)$ and $U(1)$-graviton-graviton anomalies in the theory it is necessary that the sum of the $U(1)$ quantum numbers of all chiral superfields and the sum of their cubes vanish.)

The model we consider is the supersymmetric version of QED. The field content is given by two chiral superfields Φ_+ and Φ_- with $U(1)$ quantum numbers $\pm e$ and an Abelian vector superfield. The Lagrangian is given by

$$\mathcal{L} = \int d^2\theta\, d^2\bar\theta \left(\bar\Phi_+ e^{2eV} \Phi_+ + \bar\Phi_- e^{-2eV} \Phi_- + \xi V \right)$$
$$+ \left[\int d^2\theta \left(\frac{1}{4} W^\alpha W_\alpha + m \Phi_+ \Phi_- \right) + h.c. \right]$$

Notice that we explicitly introduced the gauge coupling e by rescaling the gauge field ($V \to eV$). The scalar potential is

$$V = \frac{1}{8}\left(\xi + 2e\left(|z_+|^2 - |z_-|^2 \right) \right)^2 + m^2 \left(|z_+|^2 + |z_-|^2 \right)$$

and the equations for the auxiliary fields are

$$\begin{cases} F_\pm^+ = + m\, z_\mp \\ D = -\frac{1}{2}\left(\xi + 2e\left(|z_+|^2 - |z_-|^2 \right) \right) \end{cases}$$

Unless ξ vanishes, it is not possible to find a susy-invariant vacuum.

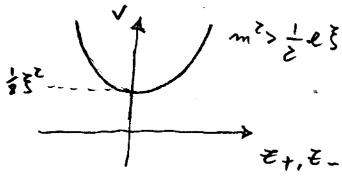
There are two regimes with different qualitative behaviours.

i) If $\underline{m^2 > \frac{1}{2} e\xi}$ the minimum of the potential occurs for $\langle z_+ \rangle = \langle z_- \rangle = 0$ and the model describes two complex scalars with masses

$$m_\pm^2 = m^2 \mp \frac{1}{2} e \xi$$

The fermion masses do not change (they are equal to m) and the gauge field and the gaugino remain massless. Notice that <u>susy is broken</u> and the gaugino plays the role of the Goldstino. <u>Gauge symmetry is unbroken</u>.

The situation is schematically described by the following potential



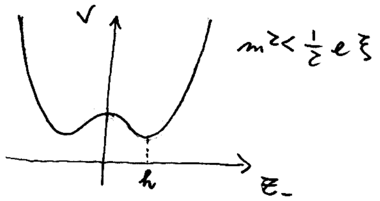
ii) If $\underline{m^2 < \frac{1}{2} e\xi}$ the minimum of the potential is at

$$\langle z_+ \rangle = 0$$
$$\langle z_- \rangle = h, \quad \text{where} \quad |h|^2 = \frac{\xi}{2e} - \frac{m^2}{e^2}.$$

There are infinite degenerate vacua corresponding to the phase of h. We can arbitrarily choose h to be real and positive, so that

$$\langle z_- \rangle = \sqrt{\frac{\xi}{2e} - \frac{m^2}{e^2}}.$$

The situation is now described by the potential



We want to find the masses of the scalars. By expanding the potential around the minimum we get

$$|z_+|^2\left(m^2+\frac{1}{2}e\xi-e^2h^2\right)+|\mathrm{Im}\,z_-|^2\left(m^2-\frac{1}{2}e\xi+e^2h^2\right)+|\mathrm{Re}\,z_-|^2\left(m^2-\frac{1}{2}e\xi+3e^2h^2\right)$$

$$=|z_+|^2(2m^2)+|\mathrm{Re}\,z_-|^2\left(e\xi-2m^2\right)$$

So we get the following masses

$$\begin{cases} m_{z_+}=\sqrt{2}\,m \\ m_{\mathrm{Re}\,z_-}=\sqrt{e\xi-2m^2}=\sqrt{2}\,eh \end{cases}$$

Notice that $\mathrm{Im}\,z_-$ has no mass term: this field is absorbed by the gauge field which becomes massive as a consequence of the gauge symmetry breaking. ($\mathrm{Im}\,z_-$ is the would-be Goldstone of the broken $U(1)$ symmetry.)

For the gauge field we have

$$(D_\mu z_-)^+ D^\mu z_- \Rightarrow e^2h^2 A_\mu A^\mu \qquad (D_\mu=\partial_\mu+ieA_\mu)$$

thus

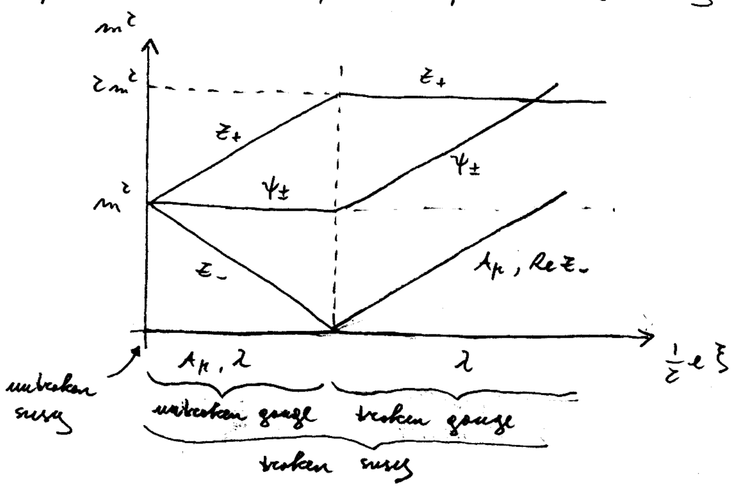
$$m_{A_\mu}=\sqrt{\frac{1}{2}e\xi-m^2}=\sqrt{2}\,eh.$$

One can also compute the fermion masses. The mass matrix for the λ and $\psi_\pm$ fermions can be diagonalized to get the eigenvectors $(\tilde\lambda,\tilde\psi_\pm)$ with masses

$$\begin{cases} m_{\tilde\lambda}=0 \\ m_{\tilde\psi_\pm}=\sqrt{e\xi-m^2}=\sqrt{2e^2h^2+m^2} \end{cases},$$

notice the presence of a Goldstino as a consequence of supersymmetry breaking.

The behaviour of the masses as a function of ξ is given by



- Mass formula

If supersymmetry is unbroken all particles within a supermultiplet have the same mass. This is no longer true if susy is spontaneously broken, but, as we will see, one can still find a relation among the masses. In the following we will analyse the mass spectrum at tree level in a generic susy theory with possible susy and/or gauge symmetry breaking.
To derive the mass matrices it is more convenient to canonically normalize the gauge fields and reintroduce the gauge couplings in the covariant derivatives. For this purpose it is sufficient to rescale the gauge superfields as $V \to gV$.

Let's start with the masses of the gauge fields. These are non-vanishing only if the gauge symmetry is broken and the vector fields, in this case, acquire an extra degree of freedom through the Higgs mechanism. The mass term comes from the term $(D_\mu z)^+ D^\mu z$, thus we get the mass term $g^2 \langle z^+ T^a T^b z \rangle v_\mu^a v^{b\mu}$ and thus the mass matrix

$$(\mathcal{M}_1^2)^{ab} = 2 g^2 \langle z^+ T^a T^b z \rangle .$$

It is convenient to introduce the notations

$$D_i^a \equiv \frac{\partial D^a}{\partial z^i} = -g\,(z^+ T^a)_i \quad , \quad D^{ia} \equiv \frac{\partial D^a}{\partial z^{i+}} = -g\,(T^a z)^i$$

$$D_j^{ai} \equiv -g\, T^{ai}_j$$

and

$$F^{ij} \equiv \frac{\partial F^j}{\partial z^{i+}} = \frac{\partial^2 W^+}{\partial z^{j+} \partial z^{i+}} \quad , \quad F_{ij} \equiv \frac{\partial F_i^+}{\partial z^j} = \frac{\partial^2 W}{\partial z^j \partial z^i} .$$

With this notations

$$\underline{(\mathcal{M}_1^2)^{ab} = 2 \langle D_i^a D^{bi} \rangle = 2 \langle D_i^a \rangle \langle D^{bi} \rangle} .$$

The mass term for the fermions is

$$-\frac{1}{2} (\psi^i \;\; \lambda^a)\, \mathcal{M}_{1/2} \begin{pmatrix} \psi^j \\ \lambda^b \end{pmatrix} + h.c. \quad , \quad \mathcal{M}_{1/2} = \begin{pmatrix} \langle F_{ij} \rangle & \sqrt{2}\, i \langle D_i^b \rangle \\ \sqrt{2}\, i \langle D_j^a \rangle & 0 \end{pmatrix}$$

with the squared masses of the fermions given by the eigenvalues of the Hermitian matrix

$$\underline{(\mathcal{M}_{1/2} \mathcal{M}_{1/2}^+) = \begin{pmatrix} \langle F_{il} \rangle \langle F^{jl} \rangle + 2 \langle D_i^c \rangle \langle D^{cj} \rangle & -\sqrt{2}\, i \langle F_{il} \rangle \langle D^{bl} \rangle \\ \sqrt{2}\, i \langle D_l^a \rangle \langle F^{jl} \rangle & 2 \langle D_l^a \rangle \langle D^{bl} \rangle \end{pmatrix}} .$$

Finally for the scalars the mass terms are

$$-\frac{1}{2} (z^i \;\; z_j^+)\, \mathcal{M}_0^2 \begin{pmatrix} z_k^+ \\ z^l \end{pmatrix}$$

with

$$\mathcal{M}_0^2 = \begin{pmatrix} \left\langle \frac{\partial^2 V}{\partial z^i \partial z_k^+} \right\rangle & \left\langle \frac{\partial^2 V}{\partial z^i \partial z^l} \right\rangle \\ \left\langle \frac{\partial^2 V}{\partial z_j^+ \partial z_k^+} \right\rangle & \left\langle \frac{\partial^2 V}{\partial z_j^+ \partial z^l} \right\rangle \end{pmatrix} .$$

We find that

$$\mathcal{M}_0^2 = \begin{pmatrix} \langle F_{ip}\rangle\langle F^{kp}\rangle + \langle D^{ak}\rangle\langle D^a_i\rangle + \langle D^a\rangle D^{a\ k}_i & \langle F^p\rangle\langle F_{i\ell p}\rangle + \langle D^a_i\rangle\langle D^a_\ell\rangle \\ \langle F_p^+\rangle\langle F^{jkp}\rangle + \langle D^{aj}\rangle\langle D^{ak}\rangle & \langle F_{\ell p}\rangle\langle F^{jp}\rangle + \langle D^{aj}\rangle\langle D^a_\ell\rangle + \langle D^a\rangle D^{aj}_\ell \end{pmatrix}.$$

We can now compute the traces of the squared mass matrices, which yield the sum of the squared masses of the fields.

$$\mathrm{tr}\,\mathcal{M}_1^2 = 2\langle D^a_i\rangle\langle D^{ai}\rangle$$

$$\mathrm{tr}\,\mathcal{M}_{1/2}\mathcal{M}_{1/2}^+ = \langle F_{i\ell}\rangle\langle F^{i\ell}\rangle + 4\langle D^a_i\rangle\langle D^{ai}\rangle$$

$$\mathrm{tr}\,\mathcal{M}_0^2 = 2\langle F_{ip}\rangle\langle F^{ip}\rangle + 2\langle D^a_i\rangle\langle D^{ai}\rangle - 2g\langle D^a\rangle\,\mathrm{tr}\,T^a$$

So we get

$$\mathrm{Str}\,\mathcal{M}^2 \equiv 3\,\mathrm{tr}\,\mathcal{M}_1^2 - 2\,\mathrm{tr}\,\mathcal{M}_{1/2}\mathcal{M}_{1/2}^+ + \mathrm{tr}\,\mathcal{M}_0^2 = -2g\langle D^a\rangle\,\mathrm{tr}\,T^a.$$

In this equation Str is the so called supertrace and is defined as the difference of the trace over the bosonic states and the trace over the fermionic states taking into account the multiplicity of the various fields. Explicitly, for a massive vector we have three degrees of freedom (in the above equation we used 3 degrees of freedom also for possible massless vector states which only have two degrees of freedom on-shell, however massless states do not give a contribution to $\mathrm{tr}\,\mathcal{M}^2$), for a fermion we have two degrees of freedom, for a real scalar one degree of freedom.

We see that if $\langle D^a\rangle = 0$ or $\mathrm{tr}\,T^a = 0$ (that is no $U(1)$ factors are present) the supertrace $\mathrm{Str}\,\mathcal{M}^2$ vanishes, stating that the sum of tree level squared masses of all bosonic degrees of freedom equals the sum for all fermionic ones.
Without susy breaking the above statement is trivially true. In the case of susy breaking this supertrace formula is still a strong constraint on the mass spectrum.

Notice that, if there are unbroken symmetries (as for example conservation of charge, color, baryon and lepton number), the mass matrices can not have elements linking particles with different values of the conserved quantum numbers, so the supertrace results hold separately for each set of conserved quantum numbers.

· The Witten index

An important concept in the determination of supersymmetry breaking is the Witten index. It is a quantity which can help to determine when supersymmetry is <u>not</u> broken.

We consider the Hilbert space of the states of a supersymmetric theory $\mathcal{H}$, we define the <u>Witten index</u> as

$$I(\beta) = \mathrm{Str}_{\mathcal{H}}\, e^{-\beta H} = \mathrm{Tr}_{\mathcal{H}}\, (-1)^F e^{-\beta H}$$

where β is a positive real number and F is the fermionic number operator (that is $(-1)^F$ is $+1$ on a bosonic state and -1 on a fermionic state). We will see that $I(\beta)$ is actually independent of β and is determined uniquely by the set of zero-energy states.

The crucial observation is that in a supersymmetric theory there are an equal number of bosonic and fermionic states with any given positive energy (see Weinberg III section 29.1. for a proof of this statement). Hence the trace of $(-1)^F e^{-\beta H}$ on any given set of state with energy $H = E > 0$ vanishes, and $I(\beta)$ receives contributions only from the $E=0$ states.

In formulae

$$I(\beta) = \sum_{E\geq 0} e^{-\beta E}\, n(E)$$

where

$$n(E) = \mathrm{Tr}_{\mathcal{H}_E} (-1)^F = n_+(E) - n_-(E)$$

is the difference between the number of bosonic and fermionic states with energy E. Given that

$$\underline{n_+(E) = n_-(E)} \qquad \text{for} \quad \underline{E>0}$$

we get

$$\underline{I(\beta) = \mathrm{Tr}_{\mathcal{H}_0} (-1)^F = n_+(0) - n_-(0)}\,,$$

and $I(\beta)$ is independent of β.

The above result implies that a non-zero value for the Witten index signals the existence of some zero-energy state which implies that supersymmetry is <u>not</u> broken. (Recall that susy is unbroken if and only if there exists a vacuum with zero energy.) In contrast a zero value for the Witten index does not allow to conclude anything, since this implies only that there are an equal number of bosonic and fermionic zero-energy states, but this number could be zero (broken susy) or non-zero (unbroken susy).

Notice that the Witten index is a kind of "topological" invariant, in the sense that it does not vary if we change the parameters of the theory. This follows from the fact that we can not create or destroy states by continuously varying the parameters, we can only change their energy; but this would not change the Witten index (even if a state with $E>0$ goes to $E=0$ it always gives an equal number of fermionic and bosonic states with $E=0$, so $n_+(0)-n_-(0)$ is unchanged).

NOTE: This is no longer true if we add a perturbation which <u>changes</u> the asymptotic behaviour of the Lagrangian for large values of the fields: this can produce or destroy states.

As a simple example of the use of the Witten index, we can consider the Wess-Zumino model with one chiral superfield and a superpotential

$$W(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{1}{6} g \Phi^3 .$$

This model does not exhibit susy breaking at tree-level, but the Witten index will allow us to conclude that susy breaking does not happen even at higher order in perturbation theory nor if we consider non-perturbative effects.

Perturbation theory is trustable if m is large and g is small. In this regime there are two zero-energy bosonic states corresponding to the configurations

i) $\langle \varepsilon \rangle = 0$

ii) $\langle \varepsilon \rangle = -\frac{2m^2}{g}$

but there are no zero-energy fermionic states (the lowest energy fermionic state is a zero-momentum one-fermion state with energy $|m|$).

Thus for large m and small g the Witten index is $I = 2$ and susy must be unbroken. Because the Witten index does not change under changes in the parameters it remains equal to 2 also when g is large and we have a strongly coupled theory, or when $m = 0$ and the two potential wells merge (in this case there are massless bosons and fermions and it is not easy to compute the Witten index directly).

Since the Witten index is not zero, supersymmetry remains strictly unbroken in the Wess-Zumino model, whatever the values of its parameters.

(For other examples and a deeper discussion see Weinberg III section 29.1.)

What happens for gauge theories? In the supersymmetric version of QED with a Fayet-Iliopoulos term we saw that susy is broken, so $I = 0$. However we can show that, without Fayet-Iliopoulos terms, a generalized version of the Witten index is non-zero, so susy is unbroken.

For a simple non-Abelian gauge theory without chiral superfields, we can again show that a generalized Witten index exists which is non-zero, so susy is unbroken. If we add massive chiral superfields to this theories the Witten index does not change, so susy is still unbroken. On the other hand there is no difficulty in finding theories with additional massless chiral superfields in which supersymmetry is broken.

(For more details on susy breaking in gauge theories and the Witten index see Weinberg III section 29.4.)

- The supersymmetric Higgs mechanism

We want now to construct a model in which only the gauge symmetry is broken. This will be the analogous of the Higgs mechanism in supersymmetric models.

We start by discussing a model with only chiral superfields with a spontaneously broken global $U(1)$ symmetry. We consider three chiral superfields Φ_0, Φ_+ and Φ_- which have quantum numbers 0, $+1$ and -1 under the $U(1)$ symmetry. The superpotential is given by

$$W(\Phi) = \frac{1}{2} m \Phi_0^2 + \mu \Phi_+ \Phi_- + \lambda \Phi_0 + g \Phi \Phi_+ \Phi_- \,,$$

and is manifestly $U(1)$ invariant. The conditions to have a supersymmetric vacuum are

$$\begin{cases} F_0^+ = \lambda + m \langle z_0 \rangle + g \langle z_+ \rangle \langle z_- \rangle = 0 \\ F_+^+ = \langle z_- \rangle \left(\mu + g \langle z_0 \rangle \right) = 0 \\ F_-^+ = \langle z_+ \rangle \left(\mu + g \langle z_0 \rangle \right) = 0 \end{cases}$$

This set of equations has two solutions :

$$i) \ \langle z_+ \rangle = \langle z_- \rangle = 0 \ , \quad \langle z_0 \rangle = -\frac{\lambda}{m}$$

$$ii) \ \langle z_+ \rangle \langle z_- \rangle = -\frac{1}{g}\left(\lambda - \frac{m\mu}{g} \right) \ , \quad \langle z_0 \rangle = -\frac{\mu}{g}$$

The first vacuum does not break the $U(1)$ global symmetry, but the second does. Notice that, in the second solution, only the product $\langle z_+ \rangle \langle z_- \rangle$ is determined, so we have a continuum set of vacua. For any solution $\langle z_+ \rangle, \langle z_- \rangle$ which satisfies the condition (ii), there exists an entire class of solutions $e^{\lambda} \langle z_+ \rangle$, $e^{-\lambda} \langle z_- \rangle$, for arbitrary <u>complex</u> λ. The ground state has a larger degeneracy than required by the initial symmetry group: this stems from the fact that the theory is invariant not only under the $U(1)$ group, but also under its complex extension.

Now we want to introduce gauge invariance in the model, so we add an Abelian vector superfield V which gauges the $U(1)$ symmetry. This changes the kinetic terms as

$$\bar{\Phi}_+ e^{2eV} \Phi_+ + \bar{\Phi}_- e^{-2eV} \bar{\Phi}_- \,.$$

We get the following equation for the D auxiliary field

$$D = -e\left(\langle z_+ \rangle^+ \langle z_+ \rangle - \langle z_- \rangle^+ \langle z_- \rangle + \xi/e \right) = 0 \ ,$$

where we added the contribution from a possible Fayet-Iliopoulos term. This extra condition can be always solved by choosing an appropriate value for the λ parameter which parametrize the vacua in the global $U(1)$ case. Notice that the D term is still invariant under $U(1)$ global symmetries $\langle z_+ \rangle \to e^{i\lambda} \langle z_+ \rangle$, $\langle z_- \rangle \to e^{-i\lambda} \langle z_- \rangle$, as well as the F conditions, so the gauged model still has a set of degenerate supersymmetric vacua. In this model the Fayet-Iliopoulos term does not induce spontaneous supersymmetry breaking. We get a mass term for the gauge field

$$2e^2\left(\langle \Xi_+\rangle^+\langle \Xi_+\rangle + \langle \Xi_-\rangle^+\langle \Xi_-\rangle\right)V^2 .$$

Notice that in this case supersymmetry is not broken, so one gets a complete massive vector supermultiplet. This is the supersymmetric version of the Higgs mechanism.

· Spontaneous breaking in extended supersymmetry.

Up to now we only considered spontaneous susy breaking in simple supersymmetry. In this section we will briefly discuss what happens in extended supersymmetry. In general the supersymmetry related to a particular Q^I_α and $\bar{Q}^I_{\dot\alpha}$ pair of generators is unbroken if the vacuum is invariant, that is

$$Q^I_\alpha |0\rangle = \bar{Q}^I_{\dot\alpha}|0\rangle = 0 .$$

We have already seen that susy breaking in simple supersymmetry is related to the vacuum energy. This is also true in extended susy because (we do not sum over I)

$$\langle 0|P_0|0\rangle = \frac{1}{4}\sum_{\alpha=\dot\alpha}\langle 0|\{Q^I_\alpha,\bar{Q}^I_{\dot\alpha}\}|0\rangle = \frac{1}{4}\left(\sum_\alpha \|Q^I_\alpha|0\rangle\|^2 + \sum_{\dot\alpha}\|\bar{Q}^I_{\dot\alpha}|0\rangle\|^2\right)$$

notice that this relation is true for each I, so the supersymmetry generated by a given pair $Q^I_\alpha, \bar{Q}^I_{\dot\alpha}$ is unbroken if and only if the vacuum energy vanishes. But this obviously implies that we can either preserve all the supersymmetry (if the vacuum energy vanishes) or break susy completely (if the vacuum energy is positive). Thus the only spontaneous susy breaking pattern is

$$N \xrightarrow[\text{breaking}]{\text{spontaneous}} N=0$$

but we can not have a theory in which susy is only partially broken, for example

$$N \longrightarrow N=1 \longrightarrow N=0 \qquad \text{is forbidden.}$$

NOTE: The above statement is not true in supergravity theories, in which we can have partial susy breaking. Moreover one can have partial susy breaking in some modified theories in which part of the supersymmetry is non-linearly realized.